No multi-graviton theories in the presence of a Dirac field

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Abstract

The cross-couplings among several massless spin-two fields (described in the free limit by a sum of Pauli-Fierz actions) in the presence of a Dirac field are investigated in the framework of the deformation theory based on local BRST cohomology. Under the hypotheses of locality, smoothness of the interactions in the coupling constant, Poincaré invariance, (background) Lorentz invariance and the preservation of the number of derivatives on each field, we prove that there are no consistent cross-interactions among different gravitons in the presence of a Dirac field. The basic features of the couplings between a single Pauli-Fierz field and a Dirac field are also emphasized.

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1 Introduction

Over the last twenty years there was a sustained effort for constructing theories involving a multiplet of spin-two fields [1, 2, 3, 4]. At the same time, various couplings of a single massless spin-two field to other fields (including

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itself) have been studied in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In this context the impossibility of cross-interactions among several Einstein gravitons under certain assumptions has recently been proved in [15] by means of a cohomological approach based on the lagrangian BRST symmetry [16, 17, 18, 19, 20]. Moreover, in [15] the impossibility of cross-interactions among different Einstein gravitons in the presence of a scalar field has also been shown.

The main aim of this paper is to investigate the cross-couplings among several massless spin-two fields (described in the free limit by a sum of Pauli-Fierz actions) in the presence of a Dirac field. More precisely, under the hypotheses of locality, smoothness of the interactions in the coupling constant, Poincaré invariance, (background) Lorentz invariance and the preservation of the number of derivatives on each field, we prove that there are no consistent cross-interactions among different gravitons in the presence of a Dirac field. This result is obtained by using the deformation technique [21] combined with the local BRST cohomology [22]. It is well-known the fact that the spin-two field in metric formulation (Einstein-Hilbert theory) cannot be coupled with a Dirac field. However, as it will be seen, if we decompose the metric like $g_{\mu\nu} = \sigma_{\mu\nu} + gh_{\mu\nu}$, where $\sigma_{\mu\nu}$ is the flat metric and g is the coupling constant, then we can indeed couple Dirac spinors to $h_{\mu\nu}$ in the space of formal series with the maximum derivative order equal to one in $h_{\mu\nu}$, such that the final results agree with the usual couplings between the spin-1/2 and the massless spin-two field in the vierbein formulation [23]. Thus, our approach envisages two different aspects. One is related to the couplings between the spin-two fields and the Dirac field, while the other focuses on proving the impossibility of cross-interactions among different gravitons via Dirac spinors. In order to make the analysis as clear as possible, we initially consider the case of the couplings between a single Pauli-Fierz field [24] and a Dirac field. In this setting we compute the interaction terms to order two in the coupling constant. Next, we prove the isomorphism between the local BRST cohomologies corresponding to the Pauli-Fierz theory and respectively to the linearized version of the vierbein formulation of the spin-two field. Since the deformation procedure is controlled by the local BRST cohomology of the free theory (in ghost number zero and one), the previous isomorphism allows us to translate the results emerging from the Pauli-Fierz formulation into the vierbein version and conversely. In this manner we obtain that the first two orders of the interacting lagrangian resulting from our setting originate

in the development of the full interacting lagrangian

$$\mathcal{L}^{(\mathrm{int})} = e\bar{\psi}\left(\mathrm{i}e_{a}{}^{\mu}\gamma^{a}D_{\mu}\psi - m\psi\right) + egM\left(\bar{\psi}\psi\right),$$

where $e_{a}{}^{\mu}$ are the vierbein fields, e is the inverse of their determinant, $e = \left(\det\left(e_{a}{}^{\mu}\right)\right)^{-1}$, D_{μ} is the full covariant derivative and $M\left(\bar{\psi}\psi\right)$ is a polynomial in $\bar{\psi}\psi$. Here and in the sequel g is the coupling constant (deformation parameter).The term $eM\left(\bar{\psi}\psi\right)$ is usually omitted in most of the textbooks on General Relativity. However, it is consistent with the gauge symmetries of the lagrangian $\mathcal{L}_2 + \mathcal{L}^{(\mathrm{int})}$, where $\mathcal{L}_2$ is the full spin-two lagrangian in the vierbein formulation. With this result at hand, we start from a finite sum of Pauli-Fierz actions and a Dirac field, and prove that there are no consistent cross-interactions between different gravitons in the presence of a Dirac field.

This paper is organized in eight sections. In Section 2 we construct the BRST symmetry of a free model with a single Pauli-Fierz field and a Dirac field. Section 3 briefly addresses the deformation procedure based on BRST symmetry. In Section 4 we compute the first two orders of the interactions between one graviton and a Dirac spinor. Section 5 is dedicated to the proof of the isomorphism between the local BRST cohomologies corresponding to the Pauli-Fierz theory and respectively to the linearized version of the vierbein formulation for the spin-two field. In Section 6 we connect the results obtained in Section 4 to those from the vierbein formulation. Section 7 is devoted to the proof of the fact that there are no consistent cross-interactions among different gravitons in the presence of a Dirac field. Section 8 exposes the main conclusions of the paper. The paper also contains two appendix sections, in which some statements from the body of the paper are proved.

2 Free model: lagrangian formulation and BRST symmetry

Our starting point is represented by a free model, whose lagrangian action is written like the sum between the action of the linearized version of Einstein-Hilbert gravity (the Pauli-Fierz action [24]) and that of a massive Dirac field

$$\begin{aligned} S_0^{\mathrm{L}}\left[h_{\mu\nu},\psi,\bar{\psi}\right] &= \int d^4x\left(-\frac{1}{2}\left(\partial_{\mu}h_{\nu\rho}\right)\left(\partial^{\mu}h^{\nu\rho}\right) + \left(\partial_{\mu}h^{\mu\rho}\right)\left(\partial^{\nu}h_{\nu\rho}\right)\right. \\ &\left. -\left(\partial_{\mu}h\right)\left(\partial_{\nu}h^{\nu\mu}\right) + \frac{1}{2}\left(\partial_{\mu}h\right)\left(\partial^{\mu}h\right) + \bar{\psi}\left(\mathrm{i}\gamma^{\mu}\left(\partial_{\mu}\psi\right) - m\psi\right)\right) \end{aligned}$$

$$\equiv \int d^4x \left(\mathcal{L}^{(\mathrm{PF})} + \mathcal{L}_0^{(\mathrm{D})}\right). \tag{1}$$

Everywhere in the paper we use the flat Minkowski metric of 'mostly plus' signature, $\sigma_{\mu\nu} = (-+++)$. In the above h denotes the trace of the Pauli-Fierz field, $h = \sigma_{\mu\nu}h^{\mu\nu}$, and the fermionic fields ψ and $\bar{\psi}$ are considered to be complex (Dirac) spinors ($\bar{\psi} = \psi^\dagger\gamma^0$). We work with the Dirac representation of the γ-matrices

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0, \quad \mu = \overline{0,3}, \tag{2}$$

where $\dagger$ signifies the operation of Hermitian conjugation. Action (1) possesses an irreducible and Abelian generating set of gauge transformations

$$\delta_\epsilon h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}, \quad \delta_\epsilon\psi = \delta_\epsilon\bar{\psi} = 0, \tag{3}$$

with ϵ_μ bosonic gauge parameters. The parantheses signify symmetrization; they are never divided by the number of terms: e.g., $\partial_{(\mu}\epsilon_{\nu)} = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$, and the minimum number of terms is always used. The same is valid with respect to the notation $[\mu\cdots\nu]$, which means antisymmetrization with respect to the indices between brackets.

In order to construct the BRST symmetry for (1) we introduce the fermionic ghosts η_μ corresponding to the gauge parameters ϵ_μ and associate antifields with the original fields and ghosts, respectively denoted by $\left\{h^{*\mu\nu}, \psi^*, \bar{\psi}^*\right\}$ and $\{\eta^{*\mu}\}$. (The statistics of the antifields is opposite to that of the correlated fields/ghosts.) The antifields of the Dirac fields are bosonic spinors, assumed to satisfy the properties

$$\left(\bar{\psi}^*\right)^\dagger \gamma^0 = -\psi^*, \quad \gamma^0 \left(\psi^*\right)^\dagger = -\bar{\psi}^*. \tag{4}$$

Since the gauge generators of the free theory under study are field independent and irreducible, it follows that the BRST differential simply decomposes into

$$s = \delta + \gamma, \tag{5}$$

where δ represents the Koszul-Tate differential, graded by the antighost number agh ($\mathrm{agh}(\delta) = -1$), and γ stands for the exterior derivative along the gauge orbits, whose degree is named pure ghost number pgh ($\mathrm{pgh}(\gamma) = 1$). These two degrees do not interfere ($\mathrm{pgh}(\delta) = 0$, $\mathrm{agh}(\gamma) = 0$). The overall degree from the BRST complex is known as the ghost number gh and is

defined like the difference between the pure ghost number and the antighost number, such that $\mathrm{gh}(\delta) = \mathrm{gh}(\gamma) = \mathrm{gh}(s) = 1$. If we make the notations

$$\Phi^{\alpha_0} = \left(h_{\mu\nu}, \psi, \bar{\psi}\right), \ \Phi^*_{\alpha_0} = \left(h^{*\mu\nu}, \psi^*, \bar{\psi}^*\right), \tag{6}$$

then, according to the standard rules of the BRST formalism, the degrees of the BRST generators are valued like

$$\mathrm{agh}\left(\Phi^{\alpha_0}\right) = \mathrm{agh}\left(\eta_\mu\right) = 0, \quad \mathrm{agh}\left(\Phi^*_{\alpha_0}\right) = 1, \quad \mathrm{agh}\left(\eta^{*\mu}\right) = 2, \tag{7}$$

$$\mathrm{pgh}\left(\Phi^{\alpha_0}\right) = 0, \quad \mathrm{pgh}\left(\eta_\mu\right) = 1, \quad \mathrm{pgh}\left(\Phi^*_{\alpha_0}\right) = \mathrm{pgh}\left(\eta^{*\mu}\right) = 0. \tag{8}$$

The actions of the differentials δ and γ on the generators from the BRST complex are given by

$$\delta h^{*\mu\nu} = 2H^{\mu\nu}, \ \delta\psi^* = -\left(m\bar{\psi} + \mathrm{i}\partial_\mu\bar{\psi}\gamma^\mu\right), \tag{9}$$

$$\delta\bar{\psi}^* = -\left(\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi\right), \ \delta\eta^{*\mu} = -2\partial_\nu h^{*\mu\nu}, \tag{10}$$

$$\delta\Phi^{\alpha_0} = 0 = \delta\eta_\mu, \tag{11}$$

$$\gamma\Phi^*_{\alpha_0} = 0 = \gamma\eta^{*\mu}, \tag{12}$$

$$\gamma h_{\mu\nu} = \partial_{(\mu}\eta_{\nu)}, \quad \gamma\psi = 0 = \gamma\bar{\psi}, \quad \gamma\eta_\mu = 0, \tag{13}$$

where $H^{\mu\nu}$ is the linearized Einstein tensor

$$H^{\mu\nu} = K^{\mu\nu} - \frac{1}{2}\sigma^{\mu\nu}K, \tag{14}$$

with $K^{\mu\nu}$ and K the linearized Ricci tensor and respectively the linearized scalar curvature, both obtained from the linearized Riemann tensor

$$\begin{aligned} K_{\mu\nu\alpha\beta} = & -\frac{1}{2}\left(\partial_\mu\partial_\alpha h_{\nu\beta} + \partial_\nu\partial_\beta h_{\mu\alpha}\right. \\ & \left.-\partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\mu\partial_\beta h_{\nu\alpha}\right), \end{aligned} \tag{15}$$

via its simple and double traces

$$K_{\mu\alpha} = \sigma^{\nu\beta}K_{\mu\nu\alpha\beta}, K = \sigma^{\mu\alpha}\sigma^{\nu\beta}K_{\mu\nu\alpha\beta}. \tag{16}$$

The BRST differential is known to have a canonical action in a structure named antibracket and denoted by the symbol $(,)$ ($s\cdot = \left(\cdot, \bar{S}\right)$), which is obtained by decreeing the fields/ghosts respectively conjugated to the corresponding antifields. The generator of the BRST symmetry is a bosonic

functional of ghost number zero, which is solution to the classical master equation $\left(\bar{S},\bar{S}\right)=0$. The full solution to the master equation for the free model under study reads as

$$\bar{S}=S_0^{\mathrm{L}}\left[h_{\mu\nu},\psi,\bar{\psi}\right]+\int d^4x\,h^{*\mu\nu}\partial_{(\mu}\eta_{\nu)}. \tag{17}$$

3 Deformation of the solution to the master equation: a brief review

We begin with a "free" gauge theory, described by a lagrangian action $S_0^{\mathrm{L}}\left[\Phi^{\alpha_0}\right]$, invariant under some gauge transformations $\delta_\epsilon\Phi^{\alpha_0}=Z^{\alpha_0}_{\ \ \alpha_1}\epsilon^{\alpha_1}$, i.e. $\frac{\delta S_0^{\mathrm{L}}}{\delta\Phi^{\alpha_0}}Z^{\alpha_0}_{\ \ \alpha_1}=0$, and consider the problem of constructing consistent interactions among the fields Φ^{α_0} such that the couplings preserve both the field spectrum and the original number of gauge symmetries. This matter is addressed by means of reformulating the problem of constructing consistent interactions as a deformation problem of the solution to the master equation corresponding to the "free" theory [21]. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If an interacting gauge theory can be consistently constructed, then the solution $\bar{S}$ to the master equation $\left(\bar{S},\bar{S}\right)=0$ associated with the "free" theory can be deformed into a solution S

$$\begin{aligned}\bar{S}\rightarrow S &= \bar{S}+gS_1+g^2S_2+\cdots\\ &= \bar{S}+g\int d^Dx\,a+g^2\int d^Dx\,b+\cdots,\end{aligned} \tag{18}$$

of the master equation for the deformed theory

$$(S,S)=0, \tag{19}$$

such that both the ghost and antifield spectra of the initial theory are preserved. The equation (19) splits, according to the various orders in the coupling constant (deformation parameter) g, into a tower of equations:

$$\left(\bar{S},\bar{S}\right) = 0, \tag{20}$$

$$2\left(S_1,\bar{S}\right) = 0, \tag{21}$$

$$2\left(S_2,\bar{S}\right)+(S_1,S_1) = 0, \tag{22}$$

$$\left(S_3,\bar{S}\right)+(S_1,S_2) = 0, \tag{23}$$
$$\vdots$$

The equation (20) is fulfilled by hypothesis. The next one requires that the first-order deformation of the solution to the master equation, S_1, is a cocycle of the "free" BRST differential $s\cdot = \left(\cdot,\bar{S}\right)$. However, only cohomologically non-trivial solutions to (21) should be taken into account, as the BRST-exact ones can be eliminated by some (in general non-linear) field redefinitions. This means that S_1 pertains to the ghost number zero cohomological space of s, $H^0(s)$, which is generically non-empty due to its isomorphism to the space of physical observables of the "free" theory. It has been shown (on behalf of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations, namely (22) and (23) and so on. However, the resulting interactions may be non-local, and there might even appear obstructions if one insists on their locality. The analysis of these obstructions can be done by means of standard cohomological techniques.

4 Consistent interactions between the spin-two field and the massive Dirac field

4.1 Standard material: $H(\gamma)$ and $H(\delta|d)$

This section is devoted to the investigation of consistent cross-couplings that can be introduced between a spin-two field and a massive Dirac field. This matter is addressed in the context of the antifield-BRST deformation procedure briefly addressed in the above and relies on computing the solutions to the equations (21)–(23), etc., with the help of the free BRST cohomology.

For obvious reasons, we consider only smooth, local, (background) Lorentz invariant and, moreover, Poincaré invariant quantities (i.e. we do not allow explicit dependence on the spacetime coordinates). The smoothness of the deformations refers to the fact that the deformed solution to the master equation (18) is smooth in the coupling constant g and reduces to the original solution (17) in the free limit $g=0$. In addition, we require the conservation of the number of derivatives on each field (this condition is frequently met in the literature; for instance, see the case of cross-interactions for a collection

of Pauli-Fierz fields [15] or the couplings between the Pauli-Fierz and the massless Rarita-Schwinger fields [14]). If we make the notation $S_1 = \int d^4x\, a$, with a a local function, then the equation (21), which we have seen that controls the first-order deformation, takes the local form

$$sa = \partial_\mu m^\mu, \quad \mathrm{gh}\,(a) = 0, \quad \varepsilon\,(a) = 0, \tag{24}$$

for some local m^μ and it shows that the non-integrated density of the first-order deformation pertains to the local cohomology of the BRST differential in ghost number zero, $a \in H^0\,(s|d)$, where d denotes the exterior spacetime differential. The solution to the equation (24) is unique up to s-exact pieces plus divergences

$$a \rightarrow a + sb + \partial_\mu n^\mu,\ \mathrm{gh}\,(b) = -1,\ \varepsilon\,(b) = 1,\ \mathrm{gh}\,(n^\mu) = 0,\ \varepsilon\,(n^\mu) = 0. \tag{25}$$

At the same time, if the general solution of (24) is found to be completely trivial, $a = sb + \partial_\mu n^\mu$, then it can be made to vanish $a = 0$.

In order to analyze the equation (24), we develop a according to the antighost number

$$a = \sum_{i=0}^{I} a_i, \quad \mathrm{agh}\,(a_i) = i, \quad \mathrm{gh}\,(a_i) = 0, \quad \varepsilon\,(a_i) = 0, \tag{26}$$

and take this decomposition to stop at some finite value I of the antighost number. The fact that I in (26) is finite can be argued like in [15]. Inserting the above expansion into the equation (24) and projecting it on the various values of the antighost number with the help of the splitting (5), we obtain the tower of equations

$$\gamma a_I = \partial_\mu \overset{(I)}{m}{}^\mu, \tag{27}$$

$$\delta a_I + \gamma a_{I-1} = \partial_\mu \overset{(I-1)}{m}{}^\mu, \tag{28}$$

$$\delta a_i + \gamma a_{i-1} = \partial_\mu \overset{(i-1)}{m}{}^\mu, \quad 1 \leq i \leq I-1, \tag{29}$$

where $\left(\overset{(i)}{m}{}^\mu\right)_{i=\overline{0,I}}$ are some local currents with $\mathrm{agh}\left(\overset{(i)}{m}{}^\mu\right) = i$. Moreover, according to the general result from [15] in the absence of the collection indices, the equation (27) can be replaced[1] in strictly positive antighost numbers by

$$\gamma a_I = 0, \quad I > 0. \tag{30}$$

[1]This is because the presence of the matter fields does not modify the general results on $H\,(\gamma)$ presented in [15].

Due to the second-order nilpotency of γ ($\gamma^2 = 0$), the solution to the equation (30) is clearly unique up to γ-exact contributions

$$a_I \rightarrow a_I + \gamma b_I, \quad \mathrm{agh}\,(b_I) = I, \quad \mathrm{pgh}\,(b_I) = I - 1, \quad \varepsilon\,(b_I) = 1. \tag{31}$$

Meanwhile, if it turns out that a_I reduces to γ-exact terms only, $a_I = \gamma b_I$, then it can be made to vanish, $a_I = 0$. The non-triviality of the first-order deformation a is thus translated at its highest antighost number component into the requirement that $a_I \in H^I(\gamma)$, where $H^I(\gamma)$ denotes the cohomology of the exterior longitudinal derivative γ in pure ghost number equal to I. So, in order to solve the equation (24) (equivalent with (30) and (28) and (29)), we need to compute the cohomology of γ, $H(\gamma)$, and, as it will be made clear below, also the local cohomology of δ in pure ghost number zero, $H(\delta|d)$.

Using the results on the cohomology of the exterior longitudinal differential for a collection of Pauli-Fierz fields [15], as well as the definitions (12) and (13), we can state that $H(\gamma)$ is generated on the one hand by $\Phi^*_{\alpha_0}$, η^*_μ, ψ, $\bar{\psi}$ and $K_{\mu\nu\alpha\beta}$ together with all of their spacetime derivatives and, on the other hand, by the ghosts η_μ and $\partial_{[\mu}\eta_{\nu]}$. So, the most general (and non-trivial), local solution to (30) can be written, up to γ-exact contributions, as

$$a_I = \alpha_I\left([\psi], \left[\bar{\psi}\right], [K_{\mu\nu\alpha\beta}], \left[\Phi^*_{\alpha_0}\right], \left[\eta^*_\mu\right]\right)\omega^I\left(\eta_\mu, \partial_{[\mu}\eta_{\nu]}\right), \tag{32}$$

where the notation $f([q])$ means that f depends on q and its derivatives up to a finite order, while ω^I denotes the elements of a basis in the space of polynomials with pure ghost number I in the corresponding ghosts and their antisymmetrized first-order derivatives. The objects α_I have the pure ghost number equal to zero and are required to fulfill the property $\mathrm{agh}\,(\alpha_I) = I$ in order to ensure that the ghost number of a_I is equal to zero. Since they have a bounded number of derivatives and a finite antighost number, α_I are actually polynomials in the linearized Riemann tensor, in the antifields, in all of their derivatives, as well as in the derivatives of the Dirac fields. The anticommuting behaviour of the Dirac spinors induces that α_I are polynomials also in the undifferentiated Dirac fields, so we conclude that these elements exhibit a polynomial character in all of their arguments. Due to their γ-closeness, $\gamma\alpha_I = 0$, α_I will be called "invariant polynomials". In zero antighost number, the invariant polynomials are polynomials in the linearized Riemann tensor $K_{\mu\nu\alpha\beta}$, in the Dirac spinors, as well as in their derivatives.

Inserting (32) in (28) we obtain that a necessary (but not sufficient) condition for the existence of (non-trivial) solutions a_{I-1} is that the invariant

polynomials α_I are (non-trivial) objects from the local cohomology of the Koszul-Tate differential $H(\delta|d)$ in pure ghost number zero and in strictly positive antighost numbers $I > 0$

$$\delta\alpha_I = \partial_\mu \overset{(I-1)^\mu}{j}, \quad \mathrm{agh}\left(\overset{(I-1)^\mu}{j}\right) = I-1, \quad \mathrm{pgh}\left(\overset{(I-1)^\mu}{j}\right) = 0. \tag{33}$$

We recall that $H(\delta|d)$ is completely trivial in both strictly positive antighost *and* pure ghost numbers (for instance, see [22], Theorem 5.4 and [25]). Using the fact that the Cauchy order of the free theory under study is equal to two together with the general results from [22], according to which the local cohomology of the Koszul-Tate differential in pure ghost number zero is trivial in antighost numbers strictly greater than its Cauchy order, we can state that

$$H_J(\delta|d) = 0 \quad \text{for all } J > 2, \tag{34}$$

where $H_J(\delta|d)$ represents the local cohomology of the Koszul-Tate differential in zero pure ghost number and in antighost number J. An interesting property of invariant polynomials for the free model under study is that if an invariant polynomial α_J, with $\mathrm{agh}(\alpha_J) = J \geq 2$, is trivial in $H_J(\delta|d)$, then it can be taken to be trivial also in $H_J^{\mathrm{inv}}(\delta|d)$, i.e.

$$\left(\alpha_J = \delta b_{J+1} + \partial_\mu \overset{(J)^\mu}{c},\ \mathrm{agh}(\alpha_J) = J \geq 2\right) \Rightarrow \alpha_J = \delta\beta_{J+1} + \partial_\mu \overset{(J)^\mu}{\gamma}, \tag{35}$$

with both β_{J+1} and $\overset{(J)^\mu}{\gamma}$ invariant polynomials. Here, $H_J^{\mathrm{inv}}(\delta|d)$ denotes the invariant characteristic cohomology (the local cohomology of the Koszul-Tate differential in the space of invariant polynomials) in antighost number J. This property is proved in [15] in the case of a collection of Pauli-Fierz fields and remains valid in the case considered here since the matter fields do not carry gauge symmetries, so we can write that

$$H_J^{\mathrm{inv}}(\delta|d) = 0 \quad \text{for all } J > 2. \tag{36}$$

For the same reason the antifields of the matter fields can bring only trivial contributions to $H_J(\delta|d)$ and $H_J^{\mathrm{inv}}(\delta|d)$ for $J \geq 2$, so the results from [15] concerning both $H_2(\delta|d)$ in pure ghost number zero and $H_2^{\mathrm{inv}}(\delta|d)$ remain valid. These cohomological spaces are still spanned by the undifferentiated antifields corresponding to the ghosts

$$H_2(\delta|d) \text{ and } H_2^{\mathrm{inv}}(\delta|d) : (\eta^{*\mu}). \tag{37}$$

In contrast to the groups $(H_J(\delta|d))_{J\geq 2}$ and $\left(H_J^{\mathrm{inv}}(\delta|d)\right)_{J\geq 2}$, which are finite-dimensional, the cohomology $H_1(\delta|d)$ in pure ghost number zero, known to be related to global symmetries and ordinary conservation laws, is infinite-dimensional since the theory is free. Moreover, $H_1(\delta|d)$ involves non-trivially the antifields of the matter fields.

The previous results on $H(\delta|d)$ and $H^{\mathrm{inv}}(\delta|d)$ in strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. More precisely, based on the formulas (33)–(36), one can successively eliminate all the pieces of antighost number strictly greater that two from the non-integrated density of the first-order deformation by adding only trivial terms, so one can take, without loss of non-trivial objects, the condition $I\leq 2$ in the decomposition (26). The proof of this statement can be realized like in [15]. In addition, the last representative is of the form (32), where the invariant polynomial is necessarily a non-trivial object from $H_2^{\mathrm{inv}}(\delta|d)$ for $I=2$, and respectively from $H_1(\delta|d)$ for $I=1$.

4.2 First-order deformation

In the case $I=2$ the non-integrated density of the first-order deformation (26) becomes

$$a=a_0+a_1+a_2. \tag{38}$$

We can further decompose a in a natural manner as a sum between three kinds of deformations

$$a=a^{(\mathrm{PF})}+a^{(\mathrm{int})}+a^{(\mathrm{Dirac})}, \tag{39}$$

where $a^{(\mathrm{PF})}$ contains only fields/ghosts/antifields from the Pauli-Fierz sector, $a^{(\mathrm{int})}$ describes the cross-interactions between the two theories (so it effectively mixes both sectors), and $a^{(\mathrm{Dirac})}$ involves only the Dirac sector. The component $a^{(\mathrm{PF})}$ is completely known (for a detailed analysis see [15]) and satisfies individually an equation of the type (24). It admits a decomposition similar to (38)

$$a^{(\mathrm{PF})}=a_0^{(\mathrm{PF})}+a_1^{(\mathrm{PF})}+a_2^{(\mathrm{PF})}, \tag{40}$$

where

$$a_2^{(\mathrm{PF})} = \frac{1}{2}\eta^{*\mu}\eta^{\nu}\partial_{[\mu}\eta_{\nu]}, \tag{41}$$

$$a_1^{(\mathrm{PF})} = h^{*\mu\rho}\left(\left(\partial_\rho\eta^\nu\right)h_{\mu\nu}-\eta^\nu\partial_{[\mu}h_{\nu]\rho}\right), \tag{42}$$

and $a_0^{(\mathrm{PF})}$ is the cubic vertex of the Einstein-Hilbert lagrangian plus a cosmological term[2]. Consequently, it follows that $a^{(\mathrm{int})}$ and $a^{(\mathrm{Dirac})}$ are subject to some separate equations

$$
\begin{aligned}
sa^{(\mathrm{int})} &= \partial_\mu m^{(\mathrm{int})\mu}, &(43)\\
sa^{(\mathrm{Dirac})} &= \partial_\mu m^{(\mathrm{Dirac})\mu}, &(44)
\end{aligned}
$$

for some local m^μ's. In the sequel we analyze the general solutions to these equations.

Since the Dirac field does not carry gauge symmetries of its own, it results that the Dirac sector can only occur in antighost number one and zero, so, without loss of generality, we take

$$
a^{(\mathrm{int})} = a_0^{(\mathrm{int})} + a_1^{(\mathrm{int})} \tag{45}
$$

in (43), where the components involved in the right-hand side of (45) are subject to the equations

$$
\begin{aligned}
\gamma a_1^{(\mathrm{int})} &= 0, &(46)\\
\delta a_1^{(\mathrm{int})} + \gamma a_0^{(\mathrm{int})} &= \partial_\mu \overset{(0)}{m}{}^{(\mathrm{int})\mu}. &(47)
\end{aligned}
$$

According to (32) in pure ghost number one and because ω^1 is spanned by

$$
\omega^{1\Delta} = \left(\eta_\mu,\ \partial_{[\mu}\eta_{\nu]}\right),
$$

we infer that the most general expression of $a_1^{(\mathrm{int})}$ as solution to the equation (46), which complies with all the general requirements imposed on the interacting theory (including the preservation of the number of derivatives on each field with respect to the free theory), is

$$
\begin{aligned}
a_1^{(\mathrm{int})} = {} & \Big(k_1\psi^*\left(\partial^\alpha\psi\right) + k_1^\dagger\left(\partial^\alpha\bar{\psi}\right)\bar{\psi}^* + k_2\psi^*\gamma^\alpha\psi \\
& + k_2^\dagger\bar{\psi}\gamma^\alpha\bar{\psi}^* + k_3\psi^*\gamma^\alpha\gamma^\mu\left(\partial_\mu\psi\right) + k_3^\dagger\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\gamma^\alpha\bar{\psi}^*\Big)\eta_\alpha \\
& + \left(k_4\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\bar{\psi}^* - k_4^\dagger\psi^*\left[\gamma^\alpha,\gamma^\beta\right]\psi\right)\partial_{[\alpha}\eta_{\beta]}.
\end{aligned} \tag{48}
$$

[2]The terms $a_2^{(\mathrm{PF})}$ and $a_1^{(\mathrm{PF})}$ given in (41) and (42) differ from the corresponding ones in [15] by a γ-exact and respectively a δ-exact contribution. However, the difference between our $a_2^{(\mathrm{PF})} + a_1^{(\mathrm{PF})}$ and the corresponding sum from [15] is a s-exact modulo d quantity. The associated component of antighost number zero, $a_0^{(\mathrm{PF})}$, is nevertheless the same in both formulations. As a consequence, the object $a^{(\mathrm{PF})}$ and the first-order deformation in [15] belong to the same cohomological class from $H^0\left(s|d\right)$.

Here, $(k_j)_{j=\overline{1,4}}$ are arbitrary complex functions of $\bar{\psi}$ and ψ. If we represent them like

$$k_j = u_j + \mathrm{i}v_j,\ j = \overline{1,4}, \tag{49}$$

with u_j and v_j real functions, then direct calculations, based on the definitions (9)–(13), lead to the elimination of some of these functions from (48). For instance, the pieces proportional with the real part of k_3 are

$$\begin{aligned} & u_3 \left(\psi^* \gamma^\alpha \gamma^\mu \left(\partial_\mu \psi\right) + \left(\partial_\mu \bar{\psi}\right) \gamma^\mu \gamma^\alpha \bar{\psi}^*\right) \eta_\alpha \\ = \ & s\left(-\mathrm{i}u_3 \psi^* \gamma^\alpha \bar{\psi}^* \eta_\alpha\right) - m\left(\left(\mathrm{i}u_3\right) \psi^* \gamma^\alpha \psi + \left(\mathrm{i}u_3\right)^\dagger \bar{\psi} \gamma^\alpha \bar{\psi}^*\right) \eta_\alpha. \end{aligned} \tag{50}$$

However, we already have in $a_1^{(\mathrm{int})}$ terms proportional with $\psi^* \gamma^\alpha \psi \eta_\alpha$ and $\bar{\psi} \gamma^\alpha \bar{\psi}^* \eta_\alpha$. So, if we set $k_2 \to k_2' = k_2 - \mathrm{i}mu_3$ in (48), then we can absorb the components proportional with u_3 into those containing k_2' and $(k_2')^\dagger$ since one can always remove the s-exact terms from $a^{(\mathrm{int})}$ appearing in (50) through a redefinition of the type (25) corresponding to $n^\mu = 0$. The above analysis leads to the fact that we can safely take

$$u_3 = 0 \tag{51}$$

in (48), without loss of independent contributions to $a_1^{(\mathrm{int})}$. Strictly speaking, one may add to $a_1^{(\mathrm{int})}$ given by (48) a term of the type $\tilde{a}_1^{(\mathrm{int})} = h^{*\mu\nu} \eta_\mu F_\nu \left(\bar{\psi}, \psi\right)$. On the one hand, we observe that by applying δ on (48), then $a_1^{(\mathrm{int})}$, if consistent, would lead to some $a_0^{(\mathrm{int})}$ which contains a single field $h_{\mu\nu}$ (or one of its first-order derivatives). On the other hand, from the expression of $\delta \tilde{a}_1^{(\mathrm{int})}$ we notice that if consistent, it would give an $\tilde{a}_0^{(\mathrm{int})}$ with two $h_{\mu\nu}$ (or one $h_{\mu\nu}$ and one of its first-order derivatives). As a consequence, $\tilde{a}_1^{(\mathrm{int})}$ must satisfy, independently of $a_1^{(\mathrm{int})}$, an equation of the type $\delta \tilde{a}_1^{(\mathrm{int})} + \gamma \tilde{a}_0^{(\mathrm{int})} = \partial_\mu \rho^\mu$. However, $\tilde{a}_1^{(\mathrm{int})}$ produces a consistent $\tilde{a}_0^{(\mathrm{int})}$ if and only if $F_\nu \left(\bar{\psi}, \psi\right) = \partial_\nu F \left(\bar{\psi}, \psi\right)$. The proof of the last statement can be found in Appendix A. Under these conditions, it is easy to see that

$$\begin{aligned} \tilde{a}_1^{(\mathrm{int})} = \ & \partial_\nu \left(h^{*\mu\nu} \eta_\mu F\left(\bar{\psi}, \psi\right)\right) - \gamma\left(\frac{1}{2} h^{*\mu\nu} h_{\mu\nu} F\left(\bar{\psi}, \psi\right)\right) \\ & + s\left(\frac{1}{2} \eta^{*\mu} \eta_\mu F\left(\bar{\psi}, \psi\right)\right), \end{aligned} \tag{52}$$

so $\tilde{a}_1^{(\mathrm{int})}$ is trivial.

In order to analyze the solution $a_0^{(\mathrm{int})}$ to the equation (47), it is useful to decompose $\delta a_1^{(\mathrm{int})}$ along the number of derivatives

$$\delta a_1^{(\mathrm{int})} = \sum_{k=0}^{2} \left(\delta a_1^{(\mathrm{int})}\right)_k , \tag{53}$$

where $\left(\delta a_1^{(\mathrm{int})}\right)_k$ denotes the piece with k-derivatives from $\delta a_1^{(\mathrm{int})}$. According to this decomposition, it follows that each $\left(\delta a_1^{(\mathrm{int})}\right)_k$ should be written in a γ-exact modulo d form, such that (47) is indeed satisfied. Using the definitions (9)–(11), we consequently obtain

$$\left(\delta a_1^{(\mathrm{int})}\right)_0 = -m\left(k_2 + k_2^{\dagger}\right)\bar{\psi}\gamma^{\alpha}\psi\eta_{\alpha}. \tag{54}$$

Due to the fact that the right-hand side of (54) contains no derivatives, it results that these terms neither reduce to a total divergence nor can not produce γ-exact terms, so they must be made to vanish

$$k_2 + k_2^{\dagger} = 0, \tag{55}$$

which is equivalent, by means of (49), with

$$u_2 = 0. \tag{56}$$

The definitions (9)–(11) and the result (51) together with (56) further lead to

$$\begin{aligned}\left(\delta a_1^{(\mathrm{int})}\right)_1 = {} & -m\left(\left(k_1 - \frac{v_2}{m} + \mathrm{i}v_3\right)\bar{\psi}\left(\partial^{\alpha}\psi\right) + \left(k_1 - \frac{v_2}{m} + \mathrm{i}v_3\right)^{\dagger}\left(\partial^{\alpha}\bar{\psi}\right)\psi\right)\eta_{\alpha} \\ & + \frac{m}{2}\left(\left(\frac{v_2}{m} + \mathrm{i}v_3\right)\left(\partial_{\alpha}\bar{\psi}\right)\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi - \left(\frac{v_2}{m} + \mathrm{i}v_3\right)^{\dagger}\bar{\psi}\left[\gamma^{\alpha},\gamma^{\beta}\right]\left(\partial_{\alpha}\psi\right)\right)\eta_{\beta} \\ & - m\left(k_4 - k_4^{\dagger}\right)\bar{\psi}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\partial_{[\alpha}\eta_{\beta]}.\end{aligned} \tag{57}$$

If we make the notations

$$U\left(\bar{\psi},\psi\right) = mk_1 - v_2 + \mathrm{i}mv_3, \tag{58}$$

$$V\left(\bar{\psi},\psi\right) = v_2 + \mathrm{i}mv_3, \tag{59}$$

then the formula (57) becomes

$$\left(\delta a_1^{(\mathrm{int})}\right)_1 = \partial^\alpha \left(-U^\dagger \bar{\psi}\psi\eta_\alpha + \frac{1}{2}V\bar{\psi}\left[\gamma_\alpha,\gamma_\beta\right]\psi\eta^\beta\right) + \gamma\left(\frac{1}{2}U^\dagger\bar{\psi}\psi h\right)$$
$$+\left(\left(U^\dagger - U\right)\bar{\psi}\left(\partial^\alpha\psi\right) + \frac{1}{2}\left(V+V^\dagger\right)\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\left(\partial_\beta\psi\right) + \frac{1}{2}\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\left(\partial_\beta V\right)\right.$$
$$\left.+\bar{\psi}\psi\left(\partial^\alpha U^\dagger\right)\right)\eta_\alpha - \left(\frac{1}{4}V + m\left(k_4 - k_4^\dagger\right)\right)\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}\eta_{\beta]}. \tag{60}$$

The right-hand side from (60) is γ-exact modulo d if the functions U and V satisfy the equations

$$\left(U^\dagger - U\right)\bar{\psi}\left(\partial^\alpha\psi\right) + \bar{\psi}\psi\left(\partial^\alpha U^\dagger\right) + \frac{1}{2}\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\left(\partial_\beta V\right)$$
$$+\frac{1}{2}\left(V+V^\dagger\right)\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\left(\partial_\beta\psi\right) = \partial_\beta P^{\beta\alpha}, \tag{61}$$

where

$$\frac{1}{2}\left(P^{\alpha\beta} - P^{\beta\alpha}\right) = -\left(\frac{1}{2}V + 2m\left(k_4 - k_4^\dagger\right)\right)\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi. \tag{62}$$

By direct computation we find that the left-hand side of (61) reduces to a total derivative if

$$-U\bar{\psi}\left(\partial^\alpha\psi\right) - U^\dagger\left(\partial^\alpha\bar{\psi}\right)\psi = \frac{1}{2}\partial_\beta\left(P^{\beta\alpha} + P^{\alpha\beta} - 2\sigma^{\alpha\beta}U^\dagger\bar{\psi}\psi\right), \tag{63}$$

$$-\frac{1}{2}V\left(\partial_\beta\bar{\psi}\right)\left[\gamma^\alpha,\gamma^\beta\right]\psi + \frac{1}{2}V^\dagger\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\left(\partial_\beta\psi\right)$$
$$= \frac{1}{2}\partial_\beta\left(P^{\beta\alpha} - P^{\alpha\beta} - V\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\right). \tag{64}$$

Now, the left-hand side from (63) is a total derivative if

$$U = U^\dagger \tag{65}$$

and, in addition, U is a polynomial in $\bar{\psi}\psi$ with real coefficients. In this situation we have that

$$-U\bar{\psi}\left(\partial^\alpha\psi\right) - U^\dagger\left(\partial^\alpha\bar{\psi}\right)\psi = \partial^\alpha W, \tag{66}$$

where the function W is defined via the relation

$$U = -\frac{dW}{d\left(\bar{\psi}\psi\right)}, \tag{67}$$

such that (63) can be written like

$$\partial_\beta \left(P^{\beta\alpha} + P^{\alpha\beta} - 2\sigma^{\alpha\beta}\left(U\bar{\psi}\psi + W\right)\right) = 0. \tag{68}$$

Since the quantity $P^{\beta\alpha} + P^{\alpha\beta} - 2\sigma^{\alpha\beta}\left(U\bar{\psi}\psi + W\right)$ contains no derivatives, from (68) we obtain that

$$P^{\beta\alpha} + P^{\alpha\beta} = 2\sigma^{\alpha\beta}\left(U\bar{\psi}\psi + W\right). \tag{69}$$

Inserting (62) in (64), we arrive at

$$\begin{aligned} & -\frac{1}{2}V\left(\partial_\beta\bar{\psi}\right)\left[\gamma^\alpha,\gamma^\beta\right]\psi + \frac{1}{2}V^\dagger\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\left(\partial_\beta\psi\right) \\ = \; & 2m\partial_\beta\left(\left(k_4 - k_4^\dagger\right)\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\right). \end{aligned} \tag{70}$$

At this stage we observe that the left-hand side of the previous formula leads to a total derivative if V is a purely imaginary constant

$$V = \text{const},\ V + V^\dagger = 0, \tag{71}$$

in which case the relation (70) takes the form

$$\partial_\beta\left(-\frac{1}{2}V\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\right) = 2m\partial_\beta\left(\left(k_4 - k_4^\dagger\right)\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\right), \tag{72}$$

and thus

$$V = -4m\left(k_4 - k_4^\dagger\right). \tag{73}$$

Relying on the last result, by means of (62) we obtain

$$P^{\alpha\beta} - P^{\beta\alpha} = 0, \tag{74}$$

such that (69) gives

$$P^{\alpha\beta} = \sigma^{\alpha\beta}\left(U\bar{\psi}\psi + W\right). \tag{75}$$

Let us analyze the results deduced so far. The relations (59) and (71) allow us to state that v_3 must be a true constant

$$v_3 = \mathrm{const}_1, \tag{76}$$

while v_2 must vanish

$$v_2 = 0. \tag{77}$$

In the meantime, the formula (73) implies that

$$v_4 = -\frac{v_3}{8}. \tag{78}$$

Using (58), (65) and (77) we arrive at

$$v_1 = -v_3, \tag{79}$$

such that

$$U = mu_1. \tag{80}$$

Introducing all the above results into (60), we infer that

$$\begin{aligned} \left(\delta a_1^{(\mathrm{int})}\right)_1 = {} & \partial_\alpha \left(W\left(\bar{\psi}\psi\right)\eta^\alpha + \frac{\mathrm{i}mv_3}{2}\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\psi\eta_\beta \right) \\ & -\gamma\left(\frac{1}{2}W\left(\bar{\psi}\psi\right)h\right). \end{aligned} \tag{81}$$

Taking into account the results (77)–(79), the pieces containing two derivatives from $\delta a_1^{(\mathrm{int})}$ can be written like

$$\begin{aligned} \left(\delta a_1^{(\mathrm{int})}\right)_2 = {} & \partial_\mu \Big(\mathrm{i}u_1\bar{\psi}\left(\gamma^\alpha\left(\partial_\alpha\psi\right)\eta^\mu - \gamma^\mu\left(\partial^\alpha\psi\right)\eta_\alpha\right) \\ & + \left(\mathrm{i}u_4 - \frac{v_3}{8}\right)\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}\eta_{\beta]} \\ & + \frac{v_3}{2}\bar{\psi}\left(\gamma^\alpha\left[\gamma^\mu,\gamma^\beta\right] - \gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\right)\left(\partial_\alpha\psi\right)\eta_\beta \Big) \\ & + \gamma\Big(-\frac{\mathrm{i}u_1}{2}\bar{\psi}\left(\gamma^\mu\left(\partial_\mu\psi\right)h - \gamma^\alpha\left(\partial^\beta\psi\right)h_{\alpha\beta}\right) \\ & - \left(\mathrm{i}u_4 - \frac{v_3}{8}\right)\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\beta]\mu} \\ & + \frac{v_3}{4}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\left(\partial_\alpha\psi\right)h_{\beta\mu} \Big) \\ & + \Big(\mathrm{i}\left(\partial_\mu u_1\right)\bar{\psi}\gamma^\mu\left(\partial^\alpha\psi\right) - \mathrm{i}\left(\partial^\alpha u_1\right)\bar{\psi}\gamma^\mu\left(\partial_\mu\psi\right) \end{aligned}$$

$$
\begin{aligned}
&+2v_3\left(\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\left(\partial^\alpha\psi\right)-\left(\partial_\mu\bar{\psi}\right)\gamma^\alpha\left(\partial^\mu\psi\right)\right)\eta_\alpha \\
&+\left(\frac{\mathrm{i}}{2}\left(u_1+16u_4+\mathrm{i}v_3\right)\bar{\psi}\gamma^\alpha\partial^\beta\psi\right. \\
&\left.-\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\left(\psi\mathrm{i}\left(\partial_\mu u_4\right)+\frac{v_3}{4}\left(\partial_\mu\psi\right)\right)\right)\partial_{[\alpha}\eta_{\beta]}.
\end{aligned}
\tag{82}
$$

In consequence, $\left(\delta a_1^{(\mathrm{int})}\right)_2$ is γ-exact modulo d if

$$
v_3 = 0, \tag{83}
$$

$$
u_1\left(\bar{\psi}\psi\right)+16u_4\left(\bar{\psi},\psi\right)+\mathrm{i}v_3 = 0, \tag{84}
$$

$$
\partial_\mu u_4\left(\bar{\psi},\psi\right) = \partial_\mu u_1\left(\bar{\psi}\psi\right)=0. \tag{85}
$$

By means of (84)–(85) we get that the functions u_1 and u_4 are some constants

$$
u_1=\mathrm{const}_2,\ u_4=\mathrm{const}_3, \tag{86}
$$

related via the formula

$$
u_4=-\frac{u_1}{16}. \tag{87}
$$

As u_1 is constant, from (67) and (80) we find that

$$
W=-u_1 m\bar{\psi}\psi, \tag{88}
$$

such that (81) becomes

$$
\left(\delta a_1^{(\mathrm{int})}\right)_1=\partial_\alpha\left(-u_1 m\bar{\psi}\psi\eta^\alpha\right)+\gamma\left(\frac{u_1}{2}m\bar{\psi}\psi h\right). \tag{89}
$$

Introducing the results (83) and (87) in (82), it follows that

$$
\begin{aligned}
\left(\delta a_1^{(\mathrm{int})}\right)_2 = {} & \partial_\mu\left(\mathrm{i}u_1\bar{\psi}\left(\gamma^\alpha\left(\partial_\alpha\psi\right)\eta^\mu-\gamma^\mu\left(\partial^\alpha\psi\right)\eta_\alpha\right.\right. \\
&\left.\left.-\frac{1}{16}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}\eta_{\beta]}\right)\right) \\
&+\gamma\left(-\frac{\mathrm{i}u_1}{2}\bar{\psi}\left(\gamma^\mu\left(\partial_\mu\psi\right)h-\gamma^\alpha\left(\partial^\beta\psi\right)h_{\alpha\beta}\right.\right. \\
&\left.\left.-\frac{1}{8}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\beta]\mu}\right)\right).
\end{aligned}
\tag{90}
$$

Putting together the formulas (49), (51), (56), (77)–(79), (83), and (86)–(87), we conclude that the most general expression of $a_1^{(\mathrm{int})}$ that produces a

consistent component in antighost number zero as solution to the equation (47) and complies with all the general requirements imposed at the beginning of this section can be expressed in terms of a single arbitrary, real constant, u_1. From now on we will denote this constant by k, such that the resulting $a_1^{(\mathrm{int})}$ becomes

$$\begin{aligned} a_1^{(\mathrm{int})} &= -k\left(\frac{1}{16}\left(\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\bar{\psi}^* - \psi^*\left[\gamma^\alpha,\gamma^\beta\right]\psi\right)\partial_{[\alpha}\eta_{\beta]}\right. \\ &\quad \left. -\left(\psi^*\left(\partial^\alpha\psi\right)+\left(\partial^\alpha\bar{\psi}\right)\bar{\psi}^*\right)\eta_\alpha\right). \end{aligned} \tag{91}$$

Then, using (47), (53), (54)–(55), (89) and (90) we find that the corresponding $a_0^{(\mathrm{int})}$ is

$$\begin{aligned} a_0^{(\mathrm{int})} &= \frac{k}{2}\left(\bar{\psi}\left(\mathrm{i}\gamma^\mu\left(\partial_\mu\psi\right)-m\psi\right)h - \mathrm{i}\bar{\psi}\gamma^\alpha\left(\partial^\beta\psi\right)h_{\alpha\beta}\right. \\ &\quad \left. -\frac{\mathrm{i}}{8}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\beta]\mu}\right)+\bar{a}_0^{(\mathrm{int})}. \end{aligned} \tag{92}$$

In the above, $\bar{a}_0^{(\mathrm{int})}$ represents the general local solution to the homogeneous equation

$$\gamma\bar{a}_0^{(\mathrm{int})} = \partial_\mu\bar{m}^{(\mathrm{int})\mu}, \tag{93}$$

for some local $\bar{m}^{(\mathrm{int})\mu}$. Such solutions correspond to $\bar{a}_1^{(\mathrm{int})} = 0$ and thus they cannot deform either the gauge algebra or the gauge transformations, but simply the lagrangian at order one in the coupling constant. There are two main types of solutions to (93). The first one corresponds to $\bar{m}^{(\mathrm{int})\mu} = 0$ and is given by gauge-invariant, non-integrated densities constructed from the original fields and their spacetime derivatives. According to (32) for both pure ghost and antighost numbers equal to zero, they are given by $\bar{a}_0'^{(\mathrm{int})} = \bar{a}_0'^{(\mathrm{int})}\left([\psi],\left[\bar{\psi}\right],[K_{\mu\nu\alpha\beta}]\right)$, up to the conditions that they effectively describe cross-couplings between the two types of fields and cannot be written in a divergence-like form. Unfortunately, this type of solutions must depend on the linearized Riemann tensor (and possibly of its derivatives) in order to provide cross-couplings, and thus would lead to terms with at least two derivatives of the Dirac spinors. So, by virtue of the derivative order assumption, they must be discarded by setting $\bar{a}_0'^{(\mathrm{int})} = 0$. The second kind of solutions is associated with $\bar{m}^{(\mathrm{int})\mu} \neq 0$ in (93), being understood that they lead to cross-interactions, cannot be written in a divergence-like form

and contain at most one derivative of the fields. Consequently, we obtain that

$$
\begin{aligned}
\gamma \bar{a}_0^{(\mathrm{int})} &= \partial_\rho \left(\frac{\partial \bar{a}_0^{(\mathrm{int})}}{\partial \left(\partial_\rho h_{\mu\nu} \right)} \partial_{(\mu} \eta_{\nu)} \right) + \frac{\delta \bar{a}_0^{(\mathrm{int})}}{\delta h_{\mu\nu}} \partial_{(\mu} \eta_{\nu)} \\
&= \partial_\rho \left(\frac{\partial \bar{a}_0^{(\mathrm{int})}}{\partial \left(\partial_\rho h_{\mu\nu} \right)} \partial_{(\mu} \eta_{\nu)} + 2 \frac{\delta \bar{a}_0^{(\mathrm{int})}}{\delta h_{\rho\mu}} \eta_\mu \right) - 2\partial_\rho \left(\frac{\delta \bar{a}_0^{(\mathrm{int})}}{\delta h_{\rho\mu}} \right) \eta_\mu . \quad (94)
\end{aligned}
$$

Thus, this $\bar{a}_0^{(\mathrm{int})}$ fulfills (93) if and only if its Euler-Lagrange derivatives with respect to the Pauli-Fierz fields satisfy the equations

$$
\partial_\rho \left(\frac{\delta \bar{a}_0^{(\mathrm{int})}}{\delta h_{\rho\mu}} \right) = 0. \quad (95)
$$

Since $\bar{a}_0^{(\mathrm{int})}$ may contain at most one derivative, it follows that the solution to (95) reads as

$$
\frac{\delta \bar{a}_0^{(\mathrm{int})}}{\delta h_{\mu\nu}} = \partial_\rho D^{\rho\mu\nu}, \quad (96)
$$

where $D^{\rho\mu\nu}$ contains no derivatives and is antisymmetric in its first two indices[3]

$$
D^{\rho\mu\nu} = -D^{\mu\rho\nu}. \quad (97)
$$

We insist on the fact that a solution of the type $\delta \bar{a}_0^{(\mathrm{int})} / \delta h_{\mu\nu} = \partial_\alpha \partial_\beta D^{\mu\alpha\nu\beta}$, with $D^{\mu\alpha\nu\beta}$ possessing the symmetry properties of the Riemann tensor, is not allowed in our case due to the hypothesis on the derivative order, and hence (96) is the most general solution to the equation (95). Moreover, from (96) we have that $D^{\rho\mu\nu}$ must be symmetric in its last two indices

$$
D^{\rho\mu\nu} = D^{\rho\nu\mu}. \quad (98)
$$

The properties (97) and (98) imply that

$$
\begin{aligned}
D^{\rho\mu\nu} &= -D^{\mu\rho\nu} = -D^{\mu\nu\rho} = D^{\nu\mu\rho} \\
&= D^{\nu\rho\mu} = -D^{\rho\nu\mu} = -D^{\rho\mu\nu}, \quad (99)
\end{aligned}
$$

[3]Strictly speaking, we might have added to the right-hand side of (96) the contribution $c\sigma^{\mu\nu}$, with c an arbitrary real constant. It would not have led to cross-interactions, but to the cosmological term ch, which has already been considered in $a^{(\mathrm{PF})}$.

and hence $D^{\rho\mu\nu}=0$. Consequently, (96) yields

$$\frac{\delta \bar{a}_0^{(\mathrm{int})}}{\delta h_{\mu\nu}}=0, \tag{100}$$

and thus we can write that $\bar{a}_0^{(\mathrm{int})}=L\left([\psi],\left[\bar{\psi}\right]\right)+\partial_\mu g^\mu\left(\psi,\bar{\psi},h_{\alpha\beta}\right)$. Since we are interested only in cross-interactions, we must set $L=0$. At this stage we remain with the trivial solutions

$$\bar{a}_0^{(\mathrm{int})}=\partial_\mu g^\mu\left(\psi,\bar{\psi},h_{\alpha\beta}\right), \tag{101}$$

which can be completely removed from the first-order deformation via a transformation of the form (25) with $b=0$. In conclusion, we can take, without loss of generality

$$\bar{a}_0^{(\mathrm{int})}=0 \tag{102}$$

in the solution (92). As a consequence of the above discussion, we can state that the antighost number zero component of $a^{(\mathrm{int})}$ reads as

$$\begin{aligned} a_0^{(\mathrm{int})} &= \frac{k}{2}\left(\bar{\psi}\left(\mathrm{i}\gamma^\mu\left(\partial_\mu\psi\right)-m\psi\right)h-\mathrm{i}\bar{\psi}\gamma^\alpha\left(\partial^\beta\psi\right)h_{\alpha\beta}\right. \\ &\quad\left.-\frac{\mathrm{i}}{8}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\beta]\mu}\right). \end{aligned} \tag{103}$$

After some computation, we find that

$$\begin{aligned} a_1^{(\mathrm{int})}+a_0^{(\mathrm{int})} &= -k\left(\frac{1}{16}\left(\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\bar{\psi}^*-\psi^*\left[\gamma^\alpha,\gamma^\beta\right]\psi\right)\partial_{[\alpha}\eta_{\beta]}\right. \\ &\quad\left.-\frac{1}{2}\left(\psi^*\left(\partial^\alpha\psi\right)+\left(\partial^\alpha\bar{\psi}\right)\bar{\psi}^*-\left(\partial^\alpha\psi^*\right)\psi-\bar{\psi}\left(\partial^\alpha\bar{\psi}^*\right)\right)\eta_\alpha\right) \\ &\quad-\frac{k}{2}\Theta^{\mu\nu}h_{\mu\nu}+s\Lambda+\partial_\mu v^\mu, \end{aligned} \tag{104}$$

where

$$\Theta^{\mu\nu}=\frac{\mathrm{i}}{2}\left(\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi-\left(\partial^{(\mu}\bar{\psi}\right)\gamma^{\nu)}\psi\right), \tag{105}$$

represents the stress-energy tensor of the Dirac field, while Λ is given by

$$\Lambda=-\frac{k}{4}\left(\psi^*\psi+\bar{\psi}\bar{\psi}^*\right)h. \tag{106}$$

Obviously, the term $s\Lambda+\partial_\mu v^\mu$ from (104) is cohomologically trivial, and hence can be discarded. Thus, the coupling between a Dirac field and one graviton

at the first order in the deformation parameter takes the form $\Theta^{\mu\nu}h_{\mu\nu}$. We cannot stress enough that is not an assumption, but follows entirely from the deformation approach developed here. However, for subsequent purposes it is useful to work with the expressions (91) and (103) of $a_0^{(\mathrm{int})}$ and $a_1^{(\mathrm{int})}$.

Finally, we analyze the component $a^{(\mathrm{Dirac})}$ from (39). As the Dirac action from (1) has no non-trivial gauge invariance, it follows that $a^{(\mathrm{Dirac})}$ can only reduce to its component of antighost number zero

$$a^{(\mathrm{Dirac})} = a_0^{(\mathrm{Dirac})}\left([\psi], [\bar{\psi}]\right), \tag{107}$$

which is automatically solution to the equation $sa^{(\mathrm{Dirac})} \equiv \gamma a_0^{(\mathrm{Dirac})} = 0$. It comes from $a_1^{(\mathrm{Dirac})} = 0$ and does not deform the gauge transformations (3), but merely modifies the Dirac action. The condition that $a_0^{(\mathrm{Dirac})}$ is of maximum derivative order equal to one is translated into

$$a_0^{(\mathrm{Dirac})} = f\left(\bar{\psi}, \psi\right) + \left(\partial_\mu \bar{\psi}\right) g_1^\mu \left(\bar{\psi}, \psi\right) + g_2^\mu \left(\bar{\psi}, \psi\right)\left(\partial_\mu \psi\right), \tag{108}$$

where f, g_1^μ and g_2^μ are polynomials in the undifferentiated spinor fields (since they anticommute). The first polynomial is a scalar (bosonic and real), while the one-tensors g_1^μ and g_2^μ are fermionic and spinor-like. They are related via the relation

$$\left(g_1^\mu\right)^\dagger \gamma_0 = g_2^\mu \tag{109}$$

in order to ensure that $a_0^{(\mathrm{Dirac})}$ is indeed a scalar.

4.3 Second-order deformation

In the previous part of the paper we have seen that the first-order deformation of the theory can be written like the sum between the first-order deformation of the master equation for the Pauli-Fierz theory $S_1^{(\mathrm{PF})}$, and the 'interacting' part $S_1^{(\mathrm{int})}$

$$S_1^{(\mathrm{int})} = \int d^4x \left(a_1^{(\mathrm{int})} + a_0^{(\mathrm{int})} + a_0^{(\mathrm{Dirac})}\right), \tag{110}$$

with $a_1^{(\mathrm{int})}$, $a_0^{(\mathrm{int})}$ and $a_0^{(\mathrm{Dirac})}$ respectively given by (91), (92) and (108).

As shown in Appendix B, the first-order deformation is consistent at order two in the coupling constant if the constant k that parametrizes both $a_1^{(\mathrm{int})}$ and $a_0^{(\mathrm{int})}$ is equal to unit

$$k = 1, \tag{111}$$

and the functions appearing in $a_0^{(\mathrm{Dirac})}$ are of the form

$$f\left(\bar{\psi},\psi\right)=M\left(\bar{\psi}\psi\right),\ g_1^{\mu}=0,\ g_2^{\mu}=0, \tag{112}$$

with $M\left(\bar{\psi}\psi\right)$ a polynomial in $\bar{\psi}\psi$. Under these circumstances, we have that $S_2=S_2^{(\mathrm{PF})}+S_2^{(\mathrm{int})}$, where $S_2^{(\mathrm{PF})}$ can be deduced from [15], and

$$\begin{aligned} S_2^{(\mathrm{int})} &= \int d^4x\left(-\frac{1}{2}\left(\psi^*\left(\partial^{\alpha}\psi\right)+\left(\partial^{\alpha}\bar{\psi}\right)\bar{\psi}^*\right)\eta^{\beta}h_{\alpha\beta}+\frac{1}{32}\left(\bar{\psi}\left[\gamma^{\alpha},\gamma^{\beta}\right]\bar{\psi}^*\right.\right.\\ &\left.-\psi^*\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\right)\left(\eta^{\sigma}\partial_{[\alpha}h_{\beta]\sigma}-\frac{1}{2}h_{\sigma[\alpha}\left(\partial_{\beta]}\eta^{\sigma}-\partial^{\sigma}\eta_{\beta]}\right)\right)\\ &-\frac{\mathrm{i}}{4}\bar{\psi}\gamma^{\mu}\left(\partial^{\nu}\psi\right)\left(hh_{\mu\nu}-\frac{3}{2}h_{\mu\sigma}h_{\nu}^{\sigma}\right)-\frac{1}{4}\left(\bar{\psi}\mathrm{i}\gamma^{\mu}\left(\partial_{\mu}\psi\right)-m\bar{\psi}\psi\right)\times\\ &\times\left(h_{\alpha\beta}h^{\alpha\beta}-\frac{1}{2}h^2\right)-\frac{\mathrm{i}}{32}\bar{\psi}\gamma^{\mu}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\left(h\partial_{[\alpha}h_{\beta]\mu}\right.\\ &\left.\left.-h_{\mu}^{\sigma}\partial_{[\alpha}h_{\beta]\sigma}+h_{\alpha}^{\sigma}\left(2\partial_{[\beta}h_{\sigma]\mu}+\partial_{\mu}h_{\beta\sigma}\right)\right)+\frac{1}{2}M\left(\bar{\psi}\psi\right)h\right). \end{aligned} \tag{113}$$

The concrete expression of $S_2^{(\mathrm{int})}$ is inferred also in Appendix B. Making use of (111)–(112), it results that $S_1^{(\mathrm{int})}$ takes the final form

$$\begin{aligned} S_1^{(\mathrm{int})} &= \int d^4x\left(-\frac{1}{16}\left(\bar{\psi}\left[\gamma^{\alpha},\gamma^{\beta}\right]\bar{\psi}^*-\psi^*\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\right)\partial_{[\alpha}\eta_{\beta]}\right.\\ &+\left(\psi^*\left(\partial^{\alpha}\psi\right)+\left(\partial^{\alpha}\bar{\psi}\right)\bar{\psi}^*\right)\eta_{\alpha}\\ &+\frac{1}{2}\left(\bar{\psi}\left(\mathrm{i}\gamma^{\mu}\left(\partial_{\mu}\psi\right)-m\psi\right)h-\mathrm{i}\bar{\psi}\gamma^{\alpha}\left(\partial^{\beta}\psi\right)h_{\alpha\beta}\right.\\ &\left.\left.-\frac{\mathrm{i}}{8}\bar{\psi}\gamma^{\mu}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\partial_{[\alpha}h_{\beta]\mu}\right)+M\left(\bar{\psi}\psi\right)\right). \end{aligned} \tag{114}$$

5 Vierbein versus Pauli-Fierz formulation of spin-two field theory

In this section we correlate the linearized versions of first- and second-order formulations of spin-two field theory via the local BRST cohomology. In view of this, we start from the first-order formulation of spin-two field theory

$$\begin{aligned} S\left[e_a^{\ \mu},\omega_{\mu ab}\right] &= -\frac{1}{\lambda}\int d^4x\left(\omega_{\nu}^{\ ab}\partial_{\mu}\left(ee_a^{\ \mu}e_b^{\ \nu}\right)-\omega_{\mu}^{\ ab}\partial_{\nu}\left(ee_a^{\ \mu}e_b^{\ \nu}\right)\right.\\ &\left.+\frac{1}{2}ee_a^{\ \mu}e_b^{\ \nu}\left(\omega_{\mu}^{\ ac}\omega_{\nu\ c}^{\ b}-\omega_{\nu}^{\ ac}\omega_{\mu\ c}^{\ b}\right)\right), \end{aligned} \tag{115}$$

where $e_a{}^\mu$ is the vierbein field and $\omega_{\mu ab}$ are the components of the spin connection, while e is the inverse of the vierbein determinant

$$e = \left(\det\left(e_a{}^\mu\right)\right)^{-1}. \tag{116}$$

In order to linearize action (115), we develop the vierbein like

$$e_a{}^\mu = \delta_a{}^\mu - \frac{\lambda}{2} f_a{}^\mu,\ e = 1 + \frac{\lambda}{2} f, \tag{117}$$

where f is the trace of $f_a{}^\mu$. Consequently, we find that the linearized form of (115) reads as (we come back to the notations μ, ν, etc. for flat indices)

$$\begin{aligned} S_0'\left[f_{\mu\nu}, \omega_{\mu\alpha\beta}\right] &= \int d^4x \left(\omega_\alpha{}^{\alpha\mu}\left(\partial_\mu f - \partial^\nu f_{\mu\nu}\right) + \frac{1}{2}\omega^{\mu\alpha\beta}\partial_{[\alpha} f_{\beta]\mu} \right. \\ &\left. -\frac{1}{2}\left(\omega_\alpha{}^{\alpha\beta}\omega^\lambda{}_{\lambda\beta} - \omega^{\mu\alpha\beta}\omega_{\alpha\mu\beta}\right)\right). \end{aligned} \tag{118}$$

We mention that the field $f_{\mu\nu}$ contains a symmetric, as well as an antisymmetric part. The above linearized action is invariant under the gauge transformations

$$\delta_\epsilon f_{\mu\nu} = \partial_\mu \epsilon_\nu + \epsilon_{\mu\nu},\ \delta_\epsilon \omega_{\mu\alpha\beta} = \partial_\mu \epsilon_{\alpha\beta,} \tag{119}$$

where the latter gauge parameters are antisymmetric, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$. Eliminating the spin connection components on their equations of motion (auxiliary fields) from (118)

$$\omega_{\mu\alpha\beta}(f) = \frac{1}{2}\left(\partial_{[\mu} f_{\alpha]\beta} - \partial_{[\mu} f_{\beta]\alpha} - \partial_{[\alpha} f_{\beta]\mu}\right), \tag{120}$$

we obtain the second-order action

$$\begin{aligned} S_0'\left[f_{\mu\nu}, \omega_{\mu\alpha\beta}(f)\right] = S_0''\left[f_{\mu\nu}\right] = -\int d^4x \left(\frac{1}{8}\left(\partial^{[\mu} f^{\nu]\alpha}\right)\left(\partial_{[\mu} f_{\nu]\alpha}\right)\right. \\ \left. +\frac{1}{4}\left(\partial^{[\mu} f^{\nu]\alpha}\right)\left(\partial_{[\mu} f_{\alpha]\nu}\right) - \frac{1}{2}\left(\partial_\mu f - \partial^\nu f_{\mu\nu}\right)\left(\partial^\mu f - \partial_\alpha f^{\mu\alpha}\right)\right), \end{aligned} \tag{121}$$

subject to the gauge invariances

$$\delta_\epsilon f_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)} + \epsilon_{\mu\nu}. \tag{122}$$

If we decompose $f_{\mu\nu}$ in its symmetric and antisymmetric parts

$$f_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu},\ h_{\mu\nu} = h_{\nu\mu},\ B_{\mu\nu} = -B_{\nu\mu}, \tag{123}$$

the action (121) becomes

$$S_0''[f_{\mu\nu}] = S_0''[h_{\mu\nu}, B_{\mu\nu}] = \int d^4x \left(-\frac{1}{2}(\partial_\mu h_{\nu\rho})(\partial^\mu h^{\nu\rho}) + (\partial_\mu h^{\mu\rho})(\partial^\nu h_{\nu\rho}) \right.$$
$$\left. (-\partial_\mu h)(\partial_\nu h^{\nu\mu}) + \frac{1}{2}(\partial_\mu h)(\partial^\mu h)\right), \tag{124}$$

while the accompanying gauge transformations are given by

$$\delta_\epsilon h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}, \ \delta_\epsilon B_{\mu\nu} = \epsilon_{\mu\nu}. \tag{125}$$

It is easy to see that the right-hand side of (124) is nothing but the Pauli-Fierz action

$$S''[h_{\mu\nu}, B_{\mu\nu}] = S_0^{(\mathrm{PF})}[h_{\mu\nu}]. \tag{126}$$

Now, we show that the local BRST cohomologies associated with the formulations (118)–(119), (124)–(125) and the Pauli-Fierz model are isomorphic. As we have previously mentioned, we pass from (118)–(119) to (124)–(125) via the elimination of the auxiliary fields $\omega_{\mu\alpha\beta}$, such that the general theorems from Section 15 of the first reference in [22] ensure the isomorphism

$$H(s'|d) \simeq H(s''|d), \tag{127}$$

with s' and s'' the BRST differentials corresponding to (118)–(119) and respectively to (124)–(125). On the other hand, we observe that the field $B_{\mu\nu}$ does not appear in (124) and is subject to a shift gauge symmetry. Thus, in any cohomological class from $H(s''|d)$ one can take a representative that is independent of $B_{\mu\nu}$, the shift ghosts and all of their antifields. This is because these variables form contractible pairs that drop out from $H(s''|d)$ (see the general results from Section 14 of the first reference in [22]). As a consequence, we have that

$$H(s''|d) \simeq H(s|d), \tag{128}$$

where s is the Pauli-Fierz BRST differential. Combining (127) and (128), we arrive at

$$H(s'|d) \simeq H(s''|d) \simeq H(s|d). \tag{129}$$

Because the local BRST cohomology (in ghost number equal to zero and one) controls the deformation procedure, it results that the last isomorphisms allow one to pass in a consistent manner from the Pauli-Fierz version to the first- and second-order ones (in vierbein formulation) during the deformation procedure.

6 Analysis of the deformed theory

It is easy to see that one can go from (124)–(125) to the Pauli-Fierz version through the partial gauge-fixing $B_{\mu\nu}=0$. This gauge-fixing is a consequence of the more general gauge-fixing condition [23]

$$\sigma_{\mu[a}e_{b]}^{\ \ \mu}=0. \tag{130}$$

In the context of this partial gauge-fixing simple computation leads to the vierbein fields and the inverse of their determinant up to the second order in the coupling constant as

$$e_a^{\ \mu}=\overset{(0)}{e}{}_a^{\ \mu}+g\overset{(1)}{e}{}_a^{\ \mu}+g^2\overset{(2)}{e}{}_a^{\ \mu}+\cdots=\delta_a^{\ \mu}-\frac{g}{2}h_a^{\ \mu}+\frac{3g^2}{8}h_a^{\ \rho}h_\rho^{\ \mu}+\cdots, \tag{131}$$

$$e=\overset{(0)}{e}+g\overset{(1)}{e}+g^2\overset{(2)}{e}+\cdots=1+\frac{g}{2}h+\frac{g^2}{8}\left(h^2-2h_{\mu\nu}h^{\mu\nu}\right)+\cdots. \tag{132}$$

Based on the isomorphisms (129), we can further pass to the analysis of the deformed theory obtained in the previous sections. The component of antighost number equal to zero in $S_1^{(\mathrm{int})}$ is precisely the interacting lagrangian at order one in the coupling constant

$$\begin{aligned}\mathcal{L}_1^{(\mathrm{int})}&=a_0^{(\mathrm{int})}+a_0^{(\mathrm{Dirac})}=\left[\frac{1}{2}\bar{\psi}\left(\mathrm{i}\gamma^\mu\left(\partial_\mu\psi\right)-m\psi\right)h\right]+\left[-\frac{\mathrm{i}}{2}\bar{\psi}\gamma^\alpha\left(\partial^\beta\psi\right)h_{\alpha\beta}\right]\\&+\left[-\frac{\mathrm{i}}{16}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\beta]\mu}\right]+\left[M\left(\bar{\psi}\psi\right)\right]\\&\equiv\overset{(1)}{e}\mathcal{L}_0^{(\mathrm{D})}+\overset{(0)}{e}\overset{(1)}{e}{}_a^{\ \mu}\bar{\psi}\mathrm{i}\gamma^a\overset{(0)}{D}_\mu\psi+\overset{(0)}{e}\overset{(0)}{e}{}_a^{\ \mu}\bar{\psi}\mathrm{i}\gamma^a\overset{(1)}{D}_\mu\psi+\overset{(0)}{e}M\left(\bar{\psi}\psi\right),\end{aligned} \tag{133}$$

where

$$\overset{(0)}{D}_\mu=\partial_\mu, \tag{134}$$

and

$$\overset{(1)}{D}_\mu=\frac{1}{16}\overset{(1)}{\omega}_{\mu ab}\left[\gamma^a,\gamma^b\right], \tag{135}$$

with

$$\overset{(1)}{\omega}_{\mu ab}=-\partial_{[a}h_{b]\mu} \tag{136}$$

the linearized form of the full spin-connection

$$\begin{aligned}\omega_{\mu ab}&=e_b^{\ \nu}\partial_\nu e_{a\mu}-e_a^{\ \nu}\partial_\nu e_{b\mu}+e_{a\nu}\partial_\mu e_b^{\ \nu}\\&\quad-e_{b\nu}\partial_\mu e_a^{\ \nu}+e_{[a}^{\ \ \rho}e_{b]}^{\ \ \nu}e_{c\mu}\partial_\nu e^c_{\ \rho}\\&=g\overset{(1)}{\omega}_{\mu ab}+g^2\overset{(2)}{\omega}_{\mu ab}+\cdots.\end{aligned} \tag{137}$$

In (137) $e_{a\mu}$ represents the inverse of the vierbein field. Along the same line, the piece of antighost number equal to zero from the second-order deformation offers us the interacting lagrangian at order two in the coupling constant

$$
\begin{aligned}
\mathcal{L}_2^{(\mathrm{int})} = b_0^{(\mathrm{int})} &= \left[\frac{1}{8}\left(\bar{\psi}\mathrm{i}\gamma^\mu\left(\partial_\mu\psi\right) - m\bar{\psi}\psi\right)\left(h^2 - 2h_{\mu\nu}h^{\mu\nu}\right)\right] + \left[\frac{1}{2}hM\left(\bar{\psi}\psi\right)\right] \\
&+ \left[-\frac{\mathrm{i}}{4}\bar{\psi}\gamma^\mu\left(\partial^\nu\psi\right)hh_{\mu\nu}\right] + \left[-\frac{\mathrm{i}}{32}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi h\partial_{[\alpha}h_{\beta]\mu}\right] \\
&+ \left[\frac{3\mathrm{i}}{8}\bar{\psi}\gamma^\mu\left(\partial^\nu\psi\right)h_{\mu\sigma}h_\nu^\sigma\right] + \left[\frac{\mathrm{i}}{32}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi h_\mu^\sigma\partial_{[\alpha}h_{\beta]\sigma}\right] \\
&+ \left[-\frac{\mathrm{i}}{64}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\left(2h_{[\alpha}^\rho\left(\partial_{\beta]}h_{\rho\mu}\right) + 2\left(\partial_\rho h_{\mu[\alpha}\right)h_{\beta]}^\rho - \left(\partial_\mu h_{\rho[\alpha}\right)h_{\beta]}^\rho\right)\right] \\
\equiv & \overset{(2)}{e}\mathcal{L}_0^{(\mathrm{D})} + \overset{(1)}{e}M\left(\bar{\psi}\psi\right) + \overset{(1)}{e}\overset{(1)}{e}{}_a{}^\mu\bar{\psi}\mathrm{i}\gamma^a\overset{(0)}{D}_\mu\psi + \overset{(1)}{e}\overset{(0)}{e}{}_a{}^\mu\bar{\psi}\mathrm{i}\gamma^a\overset{(1)}{D}_\mu\psi \\
&+ \overset{(0)}{e}\overset{(2)}{e}{}_a{}^\mu\bar{\psi}\mathrm{i}\gamma^a\overset{(0)}{D}_\mu\psi + \overset{(0)}{e}\overset{(1)}{e}{}_a{}^\mu\bar{\psi}\mathrm{i}\gamma^a\overset{(1)}{D}_\mu\psi + \overset{(0)}{e}\overset{(0)}{e}{}_a{}^\mu\bar{\psi}\mathrm{i}\gamma^a\overset{(2)}{D}_\mu\psi,
\end{aligned}
\tag{138}
$$

where

$$
\overset{(2)}{D}_\mu = \frac{1}{16}\overset{(2)}{\omega}_{\mu ab}\left[\gamma^a,\gamma^b\right], \tag{139}
$$

while

$$
\overset{(2)}{\omega}_{\mu ab} = -\frac{1}{4}\left(2h_{c[a}\left(\partial_{b]}h^c{}_\mu\right) - 2h_{[a}{}^\nu\partial_\nu h_{b]\mu} - \left(\partial_\mu h_{[a}{}^\nu\right)h_{b]\nu}\right) \tag{140}
$$

is the second-order approximation of the spin-connection. With the help of (133) and (138) we deduce that $\mathcal{L}_0^{(\mathrm{D})} + g\mathcal{L}_1^{(\mathrm{int})} + g^2\mathcal{L}_2^{(\mathrm{int})} + \cdots$ comes from expanding the fully deformed lagrangian

$$
\mathcal{L}^{(\mathrm{int})} = e\bar{\psi}\left(\mathrm{i}e_a{}^\mu\gamma^a D_\mu\psi - m\psi\right) + geM\left(\bar{\psi}\psi\right), \tag{141}
$$

where

$$
D_\mu\psi = \partial_\mu\psi + \frac{1}{16}\omega_{\mu ab}\left[\gamma^a,\gamma^b\right]\psi \tag{142}
$$

is the full covariant derivative of ψ.

The pieces linear in the antifields ψ^* and $\bar{\psi}^*$ from the deformed solution to the master equation give us the deformed gauge transformations for the Dirac fields as

$$
\delta_\epsilon\psi = g\left(\partial^\alpha\psi\right)\epsilon_\alpha + \frac{g}{16}\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}\epsilon_{\beta]} - \frac{g^2}{2}\left(\partial^\alpha\psi\right)\epsilon^\beta h_{\alpha\beta}
$$

$$
\begin{aligned}
& -\frac{g^2}{32}\left[\gamma^\alpha,\gamma^\beta\right]\psi\left(\epsilon^\sigma\partial_{[\alpha}h_{\beta]\sigma}-\frac{1}{2}h_{\sigma[\alpha}\left(\partial_{\beta]}\epsilon^\sigma-\partial^\sigma\epsilon_{\beta]}\right)\right)+\cdots \\
= \; & g\overset{(1)}{\delta}_\epsilon\psi+g^2\overset{(2)}{\delta}_\epsilon\psi+\cdots,
\end{aligned}
\tag{143}
$$

$$
\begin{aligned}
\delta_\epsilon\bar{\psi} = \; & g\left(\partial^\alpha\bar{\psi}\right)\epsilon_\alpha-\frac{g}{16}\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\partial_{[\alpha}\epsilon_{\beta]}-\frac{g^2}{2}\left(\partial^\alpha\bar{\psi}\right)\epsilon^\beta h_{\alpha\beta} \\
& +\frac{g^2}{32}\bar{\psi}\left[\gamma^\alpha,\gamma^\beta\right]\left(\epsilon^\sigma\partial_{[\alpha}h_{\beta]\sigma}-\frac{1}{2}h_{\sigma[\alpha}\left(\partial_{\beta]}\epsilon^\sigma-\partial^\sigma\epsilon_{\beta]}\right)\right)+\cdots \\
= \; & g\overset{(1)}{\delta}_\epsilon\bar{\psi}+g^2\overset{(2)}{\delta}_\epsilon\bar{\psi}+\cdots.
\end{aligned}
\tag{144}
$$

The first two orders of the gauge transformations can be put under the form

$$\overset{(1)}{\delta}_\epsilon\psi = (\partial_\mu\psi)\overset{(0)}{\epsilon}{}^\mu+\frac{1}{8}\left[\gamma^a,\gamma^b\right]\psi\overset{(0)}{\epsilon}_{ab}, \tag{145}$$

$$\overset{(2)}{\delta}_\epsilon\psi = (\partial_\mu\psi)\overset{(1)}{\epsilon}{}^\mu+\frac{1}{8}\left[\gamma^a,\gamma^b\right]\psi\overset{(1)}{\epsilon}_{ab}, \tag{146}$$

$$\overset{(1)}{\delta}_\epsilon\bar{\psi} = \left(\partial_\mu\bar{\psi}\right)\overset{(0)}{\epsilon}{}^\mu-\frac{1}{8}\bar{\psi}\left[\gamma^a,\gamma^b\right]\overset{(0)}{\epsilon}_{ab}, \tag{147}$$

$$\overset{(2)}{\delta}_\epsilon\bar{\psi} = \left(\partial_\mu\bar{\psi}\right)\overset{(1)}{\epsilon}{}^\mu-\frac{1}{8}\bar{\psi}\left[\gamma^a,\gamma^b\right]\overset{(1)}{\epsilon}_{ab}, \tag{148}$$

where we used the notations

$$\overset{(0)}{\epsilon}{}^\mu = \epsilon^\mu=\epsilon^a\delta_a{}^\mu,\quad \overset{(1)}{\epsilon}{}^\mu=-\frac{1}{2}\epsilon^a h_a{}^\mu, \tag{149}$$

$$\overset{(0)}{\epsilon}_{ab} = \frac{1}{2}\partial_{[a}\epsilon_{b]}, \tag{150}$$

$$\overset{(1)}{\epsilon}_{ab} = -\frac{1}{4}\epsilon^c\partial_{[a}h_{b]c}+\frac{1}{8}h^c_{[a}\partial_{b]}\epsilon_c+\frac{1}{8}\left(\partial_c\epsilon_{[a}\right)h^c_{b]}. \tag{151}$$

Based on these notations, the gauge transformations of the spinors take the form

$$
\begin{aligned}
\delta_\epsilon\psi = \; & g\left((\partial_\mu\psi)\left(\overset{(0)}{\epsilon}{}^\mu+g\overset{(1)}{\epsilon}{}^\mu+\cdots\right)\right. \\
& \left.+\frac{1}{8}\left[\gamma^a,\gamma^b\right]\psi\left(\overset{(0)}{\epsilon}_{ab}+g\overset{(1)}{\epsilon}_{ab}+\cdots\right)\right),
\end{aligned}
\tag{152}
$$

$$
\begin{aligned}
\delta_\epsilon \bar{\psi} &= g\left(\left(\partial_\mu \bar{\psi}\right)\left(\overset{(0)}{\bar{\epsilon}}{}^{\mu} + g \overset{(1)}{\bar{\epsilon}}{}^{\mu} + \cdots\right)\right. \\
&\quad \left. -\frac{1}{8}\bar{\psi}\left[\gamma^a, \gamma^b\right]\left(\overset{(0)}{\epsilon}{}_{ab} + g \overset{(1)}{\epsilon}{}_{ab} + \cdots\right)\right).
\end{aligned} \tag{153}
$$

The gauge parameters $\overset{(0)}{\epsilon}{}_{ab}$ si $\overset{(1)}{\epsilon}{}_{ab}$ are precisely the first two terms from the Lorentz parameters expressed in terms of the flat parameters ϵ^a via the partial gauge-fixing (130). Indeed, (130) leads to

$$
\delta_\epsilon \left(\sigma_{\mu[a} e_{b]}{}^{\mu}\right) = 0, \tag{154}
$$

where

$$
\delta_\epsilon e_a{}^{\mu} = \bar{\epsilon}^\rho \partial_\rho e_a{}^{\mu} - e_a{}^{\rho} \partial_\rho \bar{\epsilon}^\mu + \epsilon_a{}^{b} e_b{}^{\mu}. \tag{155}
$$

Substituting (131) together with the expansions

$$
\bar{\epsilon}^\mu = \overset{(0)}{\bar{\epsilon}}{}^{\mu} + g \overset{(1)}{\bar{\epsilon}}{}^{\mu} + \cdots = \left(\delta_a{}^{\mu} - \frac{g}{2} h_a{}^{\mu} + \cdots\right)\epsilon^a \tag{156}
$$

and

$$
\epsilon_{ab} = \overset{(0)}{\epsilon}{}_{ab} + g \overset{(1)}{\epsilon}{}_{ab} + \cdots \tag{157}
$$

in (154), we arrive precisely to (150)–(151). At this point it is easy to see that the gauge transformations (152)–(153) come from the perturbative expansion of the full gauge transformations

$$
\delta_\epsilon \psi = g\left(\left(\partial_\mu \psi\right)\bar{\epsilon}^\mu + \frac{1}{8}\left[\gamma^a, \gamma^b\right]\psi \epsilon_{ab}\right), \tag{158}
$$

$$
\delta_\epsilon \bar{\psi} = g\left(\left(\partial_\mu \bar{\psi}\right)\bar{\epsilon}^\mu - \frac{1}{8}\bar{\psi}\left[\gamma^a, \gamma^b\right]\epsilon_{ab}\right). \tag{159}
$$

The full gauge transformations can be suggestively written like

$$
\delta_\epsilon \psi = g\left(\left(\partial_\mu \psi\right)\bar{\epsilon}^\mu + \frac{1}{2}\Sigma^{ab}\psi \epsilon_{ab}\right), \tag{160}
$$

$$
\delta_\epsilon \bar{\psi} = g\left(\left(\partial_\mu \bar{\psi}\right)\bar{\epsilon}^\mu - \frac{1}{2}\bar{\psi}\Sigma^{ab}\epsilon_{ab}\right), \tag{161}
$$

where

$$
\Sigma^{ab} = \frac{1}{4}\left[\gamma^a, \gamma^b\right] \tag{162}
$$

are the spin operators, whose commutators read as

$$\left[\Sigma^{ab},\Sigma^{cd}\right]=\sigma^{a[c}\Sigma^{d]b}-\sigma^{b[c}\Sigma^{d]a}. \tag{163}$$

In conclusion, the interaction between a Dirac field and one spin-two field leads to the interacting lagrangian (141), while the gauge transformations of the Dirac spinors are given by (160) and (161).

7 Impossibility of cross-interactions between gravitons in the presence of the Dirac field

As it has been proved in [15], there are no direct cross-couplings that can be introduced in a finite collection of gravitons and also no intermediate cross-couplings between different gravitons in the presence of a scalar field. In this section, under the hypotheses of locality, smoothness of the interactions in the coupling constant, Poincaré invariance, (background) Lorentz invariance and the preservation of the number of derivatives on each field, we will prove that there are no intermediate cross-couplings between different gravitons in the presence of a Dirac field.

Now, we start from a sum of Pauli-Fierz actions and a Dirac action

$$\begin{aligned} S_0^{\mathrm{L}}\left[h_{\mu\nu}^A,\psi,\bar{\psi}\right] &= \int d^4x\left(-\frac{1}{2}\left(\partial_\mu h_{\nu\rho}^A\right)\left(\partial^\mu h_A^{\nu\rho}\right)+\left(\partial_\mu h_A^{\mu\rho}\right)\left(\partial^\nu h_{\nu\rho}^A\right)\right. \\ &\quad \left.-\left(\partial_\mu h^A\right)\left(\partial_\nu h_A^{\nu\mu}\right)+\frac{1}{2}\left(\partial_\mu h^A\right)\left(\partial^\mu h_A\right)\right) \\ &\quad +\int d^4x\bar{\psi}\left(\mathrm{i}\gamma^\mu\left(\partial_\mu\psi\right)-m\psi\right), \end{aligned} \tag{164}$$

where h_A is the trace of the field $h_A^{\mu\nu}$ ($h_A=\sigma_{\mu\nu}h_A^{\mu\nu}$), while $A=1,2,\cdots,n$. The gauge transformations of the action (164) are

$$\delta_\epsilon h_{\mu\nu}^A=\partial_{(\mu}\epsilon_{\nu)}^A,\ \delta_\epsilon\psi=\delta_\epsilon\bar{\psi}=0. \tag{165}$$

The BRST complex comprises the fields/ghosts

$$\phi^{\alpha_0}=\left(h_{\mu\nu}^A,\psi,\bar{\psi}\right),\ \eta_\mu^A, \tag{166}$$

and respectively the antifields

$$\phi_{\alpha_0}^*=\left(h_A^{*\mu\nu},\psi^*,\bar{\psi}^*\right),\ \eta_A^{*\mu}. \tag{167}$$

The BRST differential splits in this situation like in (5), while the actions of δ and γ on the BRST generators are defined by

$$\begin{aligned}
\delta h_A^{*\mu\nu} &= 2H_A^{\mu\nu},\ \delta\psi^* = -\left(m\bar{\psi} + \mathrm{i}\partial_\mu\bar{\psi}\gamma^\mu\right), && (168)\\
\delta\bar{\psi}^* &= -\left(\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi\right),\ \delta\eta_A^{*\mu} = -2\partial_\nu h_A^{*\mu\nu}, && (169)\\
\delta\phi^{\alpha_0} &= 0,\ \delta\eta_\mu^A = 0, && (170)\\
\gamma\phi_{\alpha_0}^* &= 0,\ \gamma\eta_A^{*\mu} = 0, && (171)\\
\gamma h_{\mu\nu}^A &= \partial_{(\mu}\eta_{\nu)}^A,\ \gamma\psi = \gamma\bar{\psi} = 0,\ \gamma\eta_\mu^A = 0, && (172)
\end{aligned}$$

where $H_A^{\mu\nu} = K_A^{\mu\nu} - \frac{1}{2}\sigma^{\mu\nu}K_A$ is the linearized Einstein tensor for the field $h_A^{\mu\nu}$. In this case the solution to the master equation reads as

$$\bar{S} = S_0^{\mathrm{L}}\left[h_{\mu\nu}^A, \psi, \bar{\psi}\right] + \int d^4x \left(h_A^{*\mu\nu}\partial_{(\mu}\eta_{\nu)}^A\right). \tag{173}$$

The first-order deformation of the solution to the master equation may be decomposed in a manner similar to the case of a single graviton

$$\alpha = \alpha^{(\mathrm{PF})} + \alpha^{(\mathrm{int})} + \alpha^{(\mathrm{Dirac})}. \tag{174}$$

The first-order deformation in the Pauli-Fierz sector, $\alpha^{(\mathrm{PF})}$, is of the form [15]

$$\alpha^{(\mathrm{PF})} = \alpha_2^{(\mathrm{PF})} + \alpha_1^{(\mathrm{PF})} + \alpha_0^{(\mathrm{PF})}, \tag{175}$$

with

$$\alpha_2^{(\mathrm{PF})} = \frac{1}{2}f_{BC}^A\eta_A^{*\mu}\eta^{B\nu}\partial_{[\mu}\eta_{\nu]}^C. \tag{176}$$

In (176), all the coefficients f_{BC}^A are constant. The condition that $\alpha_2^{(\mathrm{PF})}$ indeed produces a consistent $\alpha_1^{(\mathrm{PF})}$ implies that these constants must be symmetric in their lower indices [15][4]

$$f_{BC}^A = f_{CB}^A. \tag{177}$$

With (177) at hand, we find that

$$\alpha_1^{(\mathrm{PF})} = f_{BC}^A h_A^{*\mu\rho}\left(\left(\partial_\rho\eta^{B\nu}\right)h_{\mu\nu}^C - \eta^{B\nu}\partial_{[\mu}h_{\nu]\rho}^C\right). \tag{178}$$

[4] The term (176) differs from that corresponding to [15] through a γ-exact term, which does not affect (177).

The requirement that $\alpha_1^{(\mathrm{PF})}$ leads to a consistent $\alpha_0^{(\mathrm{PF})}$ implies that [15][5]

$$f_{ABC} = \frac{1}{3} f_{(ABC)}, \tag{179}$$

where, by definition, $f_{ABC} = \delta_{AD} f^D_{BC}$. Based on (179), we obtain that the resulting $\alpha_0^{(\mathrm{PF})}$ reads as in [15] (where this component is denoted by a_0 and f_{ABC} by a_{abc}).

If we go along exactly the same line like in the subsection 4.2, we get that $\alpha^{(\mathrm{int})} = \alpha_1^{(\mathrm{int})} + \alpha_0^{(\mathrm{int})}$, where

$$\begin{aligned} \alpha_1^{(\mathrm{int})} &= k_A \left(\psi^* \left(\partial^\alpha \psi \right) + \left(\partial^\alpha \bar{\psi} \right) \bar{\psi}^* \right) \eta^A_\alpha \\ &\quad - \frac{k_A}{16} \left(\bar{\psi} \left[\gamma^\alpha, \gamma^\beta \right] \bar{\psi}^* - \psi^* \left[\gamma^\alpha, \gamma^\beta \right] \psi \right) \partial_{[\alpha} \eta^A_{\beta]}, \end{aligned} \tag{180}$$

$$\begin{aligned} \alpha_0^{(\mathrm{int})} &= \frac{k_A}{2} \left(\bar{\psi} \left(\mathrm{i} \gamma^\mu \left(\partial_\mu \psi \right) - m \psi \right) h^A - \mathrm{i} \bar{\psi} \gamma^\alpha \left(\partial^\beta \psi \right) h^A_{\alpha\beta} \right) \\ &\quad - \frac{\mathrm{i} k_A}{16} \bar{\psi} \gamma^\mu \left[\gamma^\alpha, \gamma^\beta \right] \psi \partial_{[\alpha} h^A_{\beta]\mu}, \end{aligned} \tag{181}$$

and k_A are some real constants. Meanwhile, we find in a direct manner that

$$\alpha^{(\mathrm{Dirac})} = a_0^{(\mathrm{Dirac})}, \tag{182}$$

with $a_0^{(\mathrm{Dirac})}$ given in (108).

Let us investigate next the consistency of the first-order deformation. If we perform the notations

$$\hat{S}_1^{(\mathrm{PF})} = \int d^4x \alpha^{(\mathrm{PF})}, \tag{183}$$

$$\hat{S}_1^{(\mathrm{int})} = \int d^4x \left(\alpha^{(\mathrm{int})} + \alpha^{(\mathrm{Dirac})} \right), \tag{184}$$

$$\hat{S}_1 = \hat{S}_1^{(\mathrm{PF})} + \hat{S}_1^{(\mathrm{int})}, \tag{185}$$

then the equation $\left(\hat{S}_1, \hat{S}_1 \right) + 2s\hat{S}_2 = 0$ (expressing the consistency of the first-order deformation) equivalently splits into the equations

$$\left(\hat{S}_1^{(\mathrm{PF})}, \hat{S}_1^{(\mathrm{PF})} \right) + 2s\hat{S}_2^{(\mathrm{PF})} = 0, \tag{186}$$

$$2 \left(\hat{S}_1^{(\mathrm{PF})}, \hat{S}_1^{(\mathrm{int})} \right) + \left(\hat{S}_1^{(\mathrm{int})}, \hat{S}_1^{(\mathrm{int})} \right) + 2s\hat{S}_2^{(\mathrm{int})} = 0, \tag{187}$$

[5]The piece (178) differs from that corresponding to [15] through a δ-exact term, which does not change (179).

where $\hat{S}_2 = \hat{S}_2^{(\mathrm{PF})} + \hat{S}_2^{(\mathrm{int})}$. The equation (186) requires that the constants f_{AB}^{C} satisfy the supplementary conditions [15]

$$f_{A[B}^{D} f_{C]D}^{E} = 0, \tag{188}$$

so they are the structure constants of a finite-dimensional, commutative, symmetric and associative real algebra $\mathcal{A}$. The analysis realized in [15] shows us that such an algebras has a trivial structure (being expressed like a direct sum of some one-dimensional ideals). So we obtain that

$$f_{AB}^{C} = 0 \quad \text{if} \quad A \neq B. \tag{189}$$

Let us analyze now the equation (187). If we denote by $\hat{\Delta}^{(\mathrm{int})}$ and $\beta^{(\mathrm{int})}$ the non-integrated densities of the functionals $2\left(\hat{S}_1^{(\mathrm{PF})}, \hat{S}_1^{(\mathrm{int})}\right) + \left(\hat{S}_1^{(\mathrm{int})}, \hat{S}_1^{(\mathrm{int})}\right)$ and respectively of $\hat{S}_2^{(\mathrm{int})}$, then the equation (187) in local form becomes

$$\hat{\Delta}^{(\mathrm{int})} = -2s\beta^{(\mathrm{int})} + \partial_\mu k^\mu, \tag{190}$$

with

$$\mathrm{gh}\left(\hat{\Delta}^{(\mathrm{int})}\right) = 1,\ \mathrm{gh}\left(\beta^{(\mathrm{int})}\right) = 0,\ \mathrm{gh}\left(k^\mu\right) = 1. \tag{191}$$

The computation of $\hat{\Delta}^{(\mathrm{int})}$ reveals in our case the following decomposition along the antighost number

$$\hat{\Delta}^{(\mathrm{int})} = \hat{\Delta}_0^{(\mathrm{int})} + \hat{\Delta}_1^{(\mathrm{int})},\ \mathrm{agh}\left(\hat{\Delta}_I^{(\mathrm{int})}\right) = I,\ I = 0, 1, \tag{192}$$

with

$$\begin{aligned} \hat{\Delta}_1^{(\mathrm{int})} \;=\; & \gamma\left(k_A k_B \left(\psi^* \left(\partial^\alpha \psi\right) + \left(\partial^\alpha \bar{\psi}\right) \bar{\psi}^*\right) \eta^{A\beta} h_{\alpha\beta}^{B}\right. \\ & -\frac{1}{16} M^{\alpha\beta} \left(\left(2k_A k_B - k_D f_{AB}^{D}\right) \eta^{A\sigma} \partial_{[\alpha} h_{\beta]\sigma}^{B} - k_D f_{AB}^{D} h_{\alpha}^{A\rho} \partial_{[\beta} \eta_{\rho]}^{B}\right)\Big) \\ & + \left(k_D f_{AB}^{D} - k_A k_B\right) \left(\left(\psi^* \left(\partial^\alpha \psi\right) + \left(\partial^\alpha \bar{\psi}\right) \bar{\psi}^*\right) \eta^{A\beta} \partial_{[\alpha} \eta_{\beta]}^{B}\right. \\ & \left. -\frac{1}{16} M^{\alpha\beta} \sigma^{\mu\nu} \partial_{[\alpha} \eta_{\mu]}^{A} \partial_{[\beta} \eta_{\nu]}^{B}\right), \end{aligned} \tag{193}$$

where we used the notation

$$M^{\alpha\beta} = \bar{\psi} \left[\gamma^\alpha, \gamma^\beta\right] \bar{\psi}^* - \psi^* \left[\gamma^\alpha, \gamma^\beta\right] \psi. \tag{194}$$

The concrete form of $\hat{\Delta}_0^{(\mathrm{int})}$ is not important in what follows and therefore we will skip it. Due to the decomposition (192), we have that $\beta^{(\mathrm{int})}$ and k^{μ} from (190) can be decomposed like

$$\begin{aligned} \beta^{(\mathrm{int})} &= \beta_0^{(\mathrm{int})} + \beta_1^{(\mathrm{int})} + \beta_2^{(\mathrm{int})},\ \mathrm{agh}\left(\beta_I^{(\mathrm{int})}\right) = I,\ I = 0, 1, 2, \qquad (195) \\ k^{\mu} &= k_0^{\mu} + k_1^{\mu} + k_2^{\mu},\ \mathrm{agh}\left(k_I^{\mu}\right) = I,\ I = 0, 1, 2. \qquad (196) \end{aligned}$$

By projecting the equation (190) on various values of the antighost number, we obtain the tower of equations

$$\begin{aligned} \gamma\beta_2^{(\mathrm{int})} &= \partial_{\mu}\left(\frac{1}{2}k_2^{\mu}\right), \qquad (197) \\ \hat{\Delta}_1^{(\mathrm{int})} &= -2\left(\delta\beta_2^{(\mathrm{int})} + \gamma\beta_1^{(\mathrm{int})}\right) + \partial_{\mu}k_1^{\mu}, \qquad (198) \\ \hat{\Delta}_0^{(\mathrm{int})} &= -2\left(\delta\beta_1^{(\mathrm{int})} + \gamma\beta_0^{(\mathrm{int})}\right) + \partial_{\mu}k_0^{\mu}. \qquad (199) \end{aligned}$$

By a trivial redefinition, the equation (197) can always be replaced with

$$\gamma\beta_2^{(\mathrm{int})} = 0. \qquad (200)$$

Analyzing the expression of $\hat{\Delta}_1^{(\mathrm{int})}$ in (193) we observe that it can be expressed as in (198) if

$$\begin{aligned} \hat{\chi} &= \left(k_D f_{AB}^D - k_A k_B\right)\left(\left(\psi^*\left(\partial^{\alpha}\psi\right) + \left(\partial^{\alpha}\bar{\psi}\right)\bar{\psi}^*\right)\eta^{A\beta}\partial_{[\alpha}\eta_{\beta]}^B\right. \\ &\left.-\frac{1}{16}M^{\alpha\beta}\sigma^{\mu\nu}\partial_{[\alpha}\eta_{\mu]}^A\partial_{[\beta}\eta_{\nu]}^B\right), \qquad (201) \end{aligned}$$

can be put in the form

$$\hat{\chi} = \delta\hat{\varphi} + \gamma\hat{\omega} + \partial_{\mu}j^{\mu}. \qquad (202)$$

Assume that (202) holds. Then, by applying δ on this equation we infer

$$\delta\hat{\chi} = \gamma\left(-\delta\hat{\omega}\right) + \partial_{\mu}\left(\delta j^{\mu}\right). \qquad (203)$$

On the other hand, if we use the concrete expression (201) of $\hat{\chi}$, by direct computation we are led to

$$\begin{aligned} \delta\hat{\chi} &= \gamma\left(\delta\left(-\left(k_D f_{AB}^D - k_A k_B\right)\bar{\psi}\bar{\psi}^*\eta_{\mu}^A\left(\partial^{\mu}h^B - \partial_{\nu}h^{B\mu\nu}\right)\right)\right. \\ &+\left(k_D f_{AB}^D - k_A k_B\right)\left(\mathrm{i}\bar{\psi}\gamma_{\mu}\left(\partial^{\alpha}\psi\right)\left(\frac{1}{2}h^{A\mu\beta}\partial_{[\alpha}\eta_{\beta]}^B - \eta^{A\beta}\partial_{[\alpha}h_{\beta]}^{B\mu}\right)\right. \end{aligned}$$

$$
\begin{aligned}
&-\frac{\mathrm{i}}{8}\bar{\psi}\gamma^{\mu}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\sigma^{\rho\lambda}\partial_{[\rho}\eta^{A}_{\alpha]}\partial_{[\lambda}\eta^{B}_{\beta]}\Big)\Big)\\
&+\partial_{\mu}\left(\delta\left(\left(k_{D}f^{D}_{AB}-k_{A}k_{B}\right)\bar{\psi}\bar{\psi}^{*}\eta^{A\nu}\left(\partial^{\mu}\eta^{B}_{\nu}-\partial_{\nu}\eta^{B\mu}\right)\right)\right.\\
&+\mathrm{i}\left(k_{D}f^{D}_{AB}-k_{A}k_{B}\right)\Big(\bar{\psi}\gamma^{\mu}\left(\partial^{\alpha}\psi\right)\eta^{A\beta}\partial_{[\alpha}\eta^{B}_{\beta]}\\
&+\frac{1}{16}\bar{\psi}\gamma^{\mu}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\sigma^{\rho\lambda}\partial_{[\rho}\eta^{A}_{\alpha]}\partial_{[\lambda}\eta^{B}_{\beta]}\Big)\Big).
\end{aligned}
\tag{204}
$$

The right-hand side of (204) can be written like in the right-hand side of (203) if the following conditions are simultaneously fulfilled

$$
\begin{aligned}
&\left(k_{D}f^{D}_{AB}-k_{A}k_{B}\right)\left(\mathrm{i}\bar{\psi}\gamma_{\mu}\left(\partial^{\alpha}\psi\right)\left(\frac{1}{2}h^{A\mu\beta}\partial_{[\alpha}\eta^{B}_{\beta]}-\eta^{A\beta}\partial_{[\alpha}h^{B\mu}_{\beta]}\right)\right.\\
&\left.-\frac{\mathrm{i}}{8}\bar{\psi}\gamma^{\mu}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\sigma^{\rho\lambda}\partial_{[\rho}\eta^{A}_{\alpha]}\partial_{[\lambda}\eta^{B}_{\beta]}\right)=-\delta\hat{\omega}',
\end{aligned}
\tag{205}
$$

$$
\begin{aligned}
&+\mathrm{i}\left(k_{D}f^{D}_{AB}-k_{A}k_{B}\right)\left(\bar{\psi}\gamma^{\mu}\left(\partial^{\alpha}\psi\right)\eta^{A\beta}\partial_{[\alpha}\eta^{B}_{\beta]}\right.\\
&\left.+\frac{1}{16}\bar{\psi}\gamma^{\mu}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi\sigma^{\rho\lambda}\partial_{[\rho}\eta^{A}_{\alpha]}\partial_{[\lambda}\eta^{B}_{\beta]}\right)=\delta j'^{\mu}.
\end{aligned}
\tag{206}
$$

However, from the action of δ on the BRST generators we observe that none of $h^{A\mu\beta}$, $\partial_{[\alpha}h^{A}_{\beta]\mu}$, η^{A}_{β} and $\partial_{[\lambda}\eta^{A}_{\beta]}$ are δ-exact. In consequence, the relations (205)–(206) hold if the equations

$$
\bar{\psi}\gamma^{\mu}\left(\partial_{\alpha}\psi\right)=\delta\Omega^{\mu}_{\alpha},
\tag{207}
$$

and

$$
\bar{\psi}\gamma^{\mu}\left[\gamma_{\alpha},\gamma_{\beta}\right]\psi=\delta\Gamma^{\mu}_{\alpha\beta}
\tag{208}
$$

take place simultaneously. Let us suppose that the relations (207)–(208) are indeed satisfied. Acting with ∂_{μ} on (207)–(208) we arrive at

$$
\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\left(\partial_{\alpha}\psi\right)\right)=\delta\left(\partial_{\mu}\Omega^{\mu}_{\alpha}\right),
\tag{209}
$$

and

$$
\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\left[\gamma_{\alpha},\gamma_{\beta}\right]\psi\right)=\delta\left(\partial_{\mu}\Gamma^{\mu}_{\alpha\beta}\right).
\tag{210}
$$

On the other hand, by direct computation we arrive at

$$
\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\left(\partial^{\alpha}\psi\right)\right)=\delta\left(-\mathrm{i}\left(\psi^{*}\left(\partial^{\alpha}\psi\right)-\bar{\psi}\left(\partial^{\alpha}\bar{\psi}^{*}\right)\right)\right),
\tag{211}
$$

$$
\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\left[\gamma_{\alpha},\gamma_{\beta}\right]\psi\right)=\delta\left(\mathrm{i}M_{\alpha\beta}\right)-4\bar{\psi}\gamma_{[\alpha}\partial_{\beta]}\psi.
\tag{212}
$$

The right-hand sides of (211)–(212) are not of the same type like the corresponding ones in (209)–(210). This means that the relations (207)–(208) are not valid, and therefore neither are (205)–(206). As a consequence, $\hat{\chi}$ must vanish, which further implies that

$$k_D f^D_{AB} - k_A k_B = 0. \tag{213}$$

Using (213) and (189) we obtain that for $A \neq B$

$$k_A k_B = 0, \tag{214}$$

which shows that the Dirac fields can couple to only one graviton, which proves the assertion from the beginning of this section.

8 Conclusion

To conclude with, in this paper we have investigated the couplings between a collection of massless spin-two fields (described in the free limit by a sum of Pauli-Fierz actions) and a Dirac field using the powerful setting based on local BRST cohomology. Initially, we have shown that, if we decompose the metric like $g_{\mu\nu} = \sigma_{\mu\nu} + g h_{\mu\nu}$, then we can couple Dirac spinors to $h_{\mu\nu}$ in the space of formal series with the maximum derivative order equal to one in $h_{\mu\nu}$, such that the final results agree with the usual couplings between the spin-1/2 and the massless spin-two field in the vierbein formulation. Based on this result, we have proved, under the hypotheses of locality, smoothness of the interactions in the coupling constant, Poincaré invariance, (background) Lorentz invariance and the preservation of the number of derivatives on each field, that there are no consistent cross-interactions among different gravitons in the presence of a Dirac field.

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A Proof of a statement made in subsection 4.2

Here, we prove that a term of the type

$$\tilde{a}_1^{(\mathrm{int})} = h^{*\mu\nu}\eta_\mu F_\nu\left(\bar{\psi},\psi\right) \tag{215}$$

is consistent in antighost number zero,

$$\delta\tilde{a}_1^{(\mathrm{int})} + \gamma\tilde{a}_0^{(\mathrm{int})} = \partial_\mu\rho^\mu, \tag{216}$$

if and only if

$$F_\nu\left(\bar{\psi},\psi\right) = \partial_\nu F\left(\bar{\psi},\psi\right). \tag{217}$$

Indeed, by applying δ on $\tilde{a}_1^{(\mathrm{int})}$ we obtain that

$$\delta\tilde{a}_1^{(\mathrm{int})} = -2H^{\mu\nu}\eta_\mu F_\nu\left(\bar{\psi},\psi\right). \tag{218}$$

It is easy to see that, if $F_\nu\left(\bar{\psi},\psi\right)$ if the form (217), then (218) implies

$$\delta\tilde{a}_1^{(\mathrm{int})} = \gamma\left(H^{\mu\nu}h_{\mu\nu}F\left(\bar{\psi},\psi\right)\right) + \partial_\mu\left(-2H^{\mu\nu}\eta_\nu F\left(\bar{\psi},\psi\right)\right), \tag{219}$$

and therefore $\tilde{a}_1^{(\mathrm{int})}$ indeed checks an equation of the type (216). Let us suppose now that $\tilde{a}_1^{(\mathrm{int})}$ satisfies the equation (216). Inserting the relations

$$\gamma\tilde{a}_0^{(\mathrm{int})} = 2\frac{\delta\tilde{a}_0^{(\mathrm{int})}}{\delta h_{\mu\nu}}\partial_\mu\eta_\nu + \partial_\mu t^\mu, \tag{220}$$

and (218) in (216), we get that

$$-2H^{\mu\nu}\eta_\mu F_\nu\left(\bar{\psi},\psi\right) + 2\frac{\delta\tilde{a}_0^{(\mathrm{int})}}{\delta h_{\mu\nu}}\partial_\nu\eta_\mu = \partial_\mu p^\mu. \tag{221}$$

The left-hand side of the last relation reduces to a total derivative if

$$H^{\mu\nu}F_\nu\left(\bar{\psi},\psi\right) = -\partial_\nu\frac{\delta\tilde{a}_0^{(\mathrm{int})}}{\delta h_{\mu\nu}}. \tag{222}$$

In order to investigate under what conditions the left-hand side of (222) also provides a total derivative, we start from the fact that

$$H^{\mu\nu} = \partial_\alpha\partial_\beta\phi^{\mu\alpha\nu\beta}, \tag{223}$$

where

$$\phi^{\mu\alpha\nu\beta} = \frac{1}{2}\left(-h^{\mu\nu}\sigma^{\alpha\beta} + h^{\alpha\nu}\sigma^{\mu\beta} + h^{\mu\beta}\sigma^{\alpha\nu} - h^{\alpha\beta}\sigma^{\mu\nu} + h\left(\sigma^{\mu\nu}\sigma^{\alpha\beta} - \sigma^{\mu\beta}\sigma^{\alpha\nu}\right)\right). \tag{224}$$

By means of (223) we further deduce that

$$H^{\mu\nu}F_{\nu} = \partial_{\nu}\left(\partial_{\beta}\phi^{\mu\nu\alpha\beta}F_{\alpha} - \phi^{\mu\beta\alpha\nu}\partial_{\beta}F_{\alpha}\right) + \frac{1}{2}\phi^{\mu\alpha\nu\beta}\partial_{\alpha}\partial_{[\beta}F_{\nu]}. \tag{225}$$

Thus, the right-hand side of (225) gives a total derivative if and only if

$$\phi^{\mu\alpha\nu\beta}\partial_{\alpha}\partial_{[\beta}F_{\nu]} = 0,$$

which further yields $F_{\nu} = \partial_{\nu}F$. This completes the proof.

B Complete computation of the second-order deformation

In this appendix we are interested in determining the complete expression of the second-order deformation for the master equation, which is known to be subject to the equation (22). Proceeding in the same manner like during the first-order deformation procedure, we can write the second-order deformation of the master equation like the sum between the Pauli-Fierz and the interacting parts

$$S_2 = S_2^{(\mathrm{PF})} + S_2^{(\mathrm{int})}. \tag{226}$$

The piece $S_2^{(\mathrm{PF})}$ describes the second-order deformation in the Pauli-Fierz sector and we will not insist on it since we are merely interested in the cross-couplings. The term $S_2^{(\mathrm{int})}$ results as solution to the equation

$$\frac{1}{2}\left(S_1, S_1\right)^{(\mathrm{int})} + sS_2^{(\mathrm{int})} = 0, \tag{227}$$

where

$$\left(S_1, S_1\right)^{(\mathrm{int})} = \left(S_1^{(\mathrm{int})}, S_1^{(\mathrm{int})}\right) + 2\left(S_1^{(\mathrm{PF})}, S_1^{(\mathrm{int})}\right). \tag{228}$$

If we denote by $\Delta^{(\mathrm{int})}$ and $b^{(\mathrm{int})}$ the non-integrated densities of $(S_1, S_1)^{(\mathrm{int})}$ and respectively of $S_2^{(\mathrm{int})}$, the local form of (227) becomes

$$\Delta^{(\mathrm{int})} = -2sb^{(\mathrm{int})} + \partial_\mu n^\mu, \tag{229}$$

with

$$\mathrm{gh}\left(\Delta^{(\mathrm{int})}\right) = 1, \quad \mathrm{gh}\left(b^{(\mathrm{int})}\right) = 0, \quad \mathrm{gh}\left(n^\mu\right) = 1, \tag{230}$$

for some local currents n^μ. Direct computation shows that $\Delta^{(\mathrm{int})}$ decomposes like

$$\Delta^{(\mathrm{int})} = \Delta_0^{(\mathrm{int})} + \Delta_1^{(\mathrm{int})}, \quad \mathrm{agh}\left(\Delta_I^{(\mathrm{int})}\right) = I, \quad I = 0, 1, \tag{231}$$

with

$$\begin{aligned}\Delta_1^{(\mathrm{int})} &= \gamma\left(k^2\left(\psi^*\left(\partial^\alpha\psi\right) + \left(\partial^\alpha\bar{\psi}\right)\bar{\psi}^*\right)\eta^\beta h_{\alpha\beta}\right. \\ &\quad \left.+\frac{k}{16}M^{\alpha\beta}\left(\left(1-2k\right)\eta^\sigma\partial_{[\alpha}h_{\beta]\sigma} + \frac{1}{2}h_{\sigma[\alpha}\left(\partial_{\beta]}\eta^\sigma - \partial^\sigma\eta_{\beta]}\right)\right)\right) \\ &\quad +k\left(1-k\right)\left(\psi^*\left(\partial_\alpha\psi\right) + \left(\partial_\alpha\bar{\psi}\right)\bar{\psi}^*\right)\eta_\beta\partial^{[\alpha}\eta^{\beta]} \\ &\quad -\frac{k}{16}\left(1-k\right)M^{\alpha\beta}\sigma^{\mu\nu}\partial_{[\alpha}\eta_{\mu]}\partial_{[\beta}\eta_{\nu]},\end{aligned} \tag{232}$$

and

$$\begin{aligned}\Delta_0^{(\mathrm{int})} &= k\mathcal{L}_0^{(\mathrm{D})}\left(h_{\alpha\beta}\partial^\alpha\eta^\beta + \eta^\beta\left(\partial_\beta h - \partial^\alpha h_{\alpha\beta}\right)\right) \\ &\quad +\frac{\mathrm{i}k}{2}\bar{\psi}\gamma^{(\alpha}\left(\partial^{\beta)}\psi\right)\left(-h_{\sigma\alpha}\partial_\beta\eta^\sigma + \eta^\sigma\left(\partial_{(\alpha}h_{\beta)\sigma} - 2\partial_\sigma h_{\alpha\beta}\right)\right) \\ &\quad +\frac{\mathrm{i}k}{8}\bar{\psi}\gamma^\mu[\gamma^\alpha,\gamma^\beta]\psi\partial_\alpha\left(\eta^\sigma\left(\partial_{(\beta}h_{\mu)\sigma} - 2\partial_\sigma h_{\mu\beta}\right) - \left(\partial_{(\beta}\eta^\sigma\right)h_{\mu)\sigma}\right) \\ &\quad +k^2h\left(\left(\partial^\sigma\mathcal{L}_0^{(\mathrm{D})}\right)\eta_\sigma + \bar{\psi}\mathrm{i}\gamma^\alpha\left(\partial^\beta\psi\right)\partial_\alpha\eta_\beta\right) \\ &\quad -\mathrm{i}k^2h_{\alpha\beta}\left(\left(\partial^\sigma\bar{\psi}\right)\gamma^\alpha\left(\partial^\beta\psi\right)\eta_\sigma + \bar{\psi}\gamma^\alpha\partial^\beta\left(\partial^\sigma\psi\eta_\sigma\right)\right) \\ &\quad -\frac{\mathrm{i}k^2}{8}\left(\partial^\sigma\left(\bar{\psi}\gamma^\mu[\gamma^\alpha,\gamma^\beta]\psi\right)\right)\eta_\sigma\partial_{[\alpha}h_{\beta]\mu} + \frac{\mathrm{i}k^2}{16}\bar{\psi}\gamma^\mu[\gamma^\alpha,\gamma^\beta]\psi h\partial_\mu\left(\partial_{[\alpha}\eta_{\beta]}\right) \\ &\quad +\frac{\mathrm{i}k^2}{16}\left(\bar{\psi}[\gamma^\mu,\gamma^\nu]\gamma^\alpha\left(\partial^\beta\psi\right)h_{\alpha\beta}\partial_{[\mu}\eta_{\nu]} - \bar{\psi}\gamma^\alpha[\gamma^\mu,\gamma^\nu]\left(\partial^\beta\left(\psi\partial_{[\mu}\eta_{\nu]}\right)\right)h_{\alpha\beta}\right) \\ &\quad +\frac{\mathrm{i}k^2}{128}\bar{\psi}\left([\gamma^\rho,\gamma^\lambda]\gamma^\mu[\gamma^\alpha,\gamma^\beta] - \gamma^\mu[\gamma^\alpha,\gamma^\beta][\gamma^\rho,\gamma^\lambda]\right)\psi\left(\partial_{[\alpha}h_{\beta]\mu}\right)\partial_{[\rho}\eta_{\lambda]} \\ &\quad +\gamma\left(-k\left(f\left(\bar{\psi},\psi\right) + \left(\partial_\mu\bar{\psi}\right)g_1^\mu\left(\bar{\psi},\psi\right) + g_2^\mu\left(\bar{\psi},\psi\right)\left(\partial_\mu\psi\right)\right)h\right.\end{aligned}$$

$$
\begin{aligned}
&+kh_{\mu\nu}\left(\left(\partial^{\nu}\bar{\psi}\right)g_1^{\mu}\left(\bar{\psi},\psi\right)+g_2^{\mu}\left(\bar{\psi},\psi\right)\left(\partial^{\nu}\psi\right)\right)\\
&+\frac{k}{8}\left(\frac{\partial^R f}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L f}{\partial\bar{\psi}}\right)\partial_{[\mu}\eta_{\nu]}\\
&+\frac{k}{8}\partial_{[\mu}\eta_{\nu]}\left(\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\left(\partial_{\rho}g_1^{\rho}\right)-\left(\partial_{\rho}g_2^{\rho}\right)\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\right.\\
&+\frac{\partial^R\left(\left(\partial_{\rho}\bar{\psi}\right)g_1^{\rho}+g_2^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\\
&-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L\left(\left(\partial_{\rho}\bar{\psi}\right)g_1^{\rho}+g_2^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\bar{\psi}}\\
&\left.-4\left(\left(\partial^{[\mu}\bar{\psi}\right)g_1^{\nu]}-g_2^{[\mu}\left(\partial^{\nu]}\psi\right)\right)\right),
\end{aligned}
\tag{233}
$$

where we used the notation

$$
M^{\alpha\beta}=\bar{\psi}\left[\gamma^{\alpha},\gamma^{\beta}\right]\bar{\psi}^{*}-\psi^{*}\left[\gamma^{\alpha},\gamma^{\beta}\right]\psi. \tag{234}
$$

Since the first-order deformation in the interacting sector starts in antighost number one, we can take, without loss of generality, the corresponding second-order deformation to start in antighost number two

$$
\begin{aligned}
b^{(\mathrm{int})} &= b_0^{(\mathrm{int})}+b_1^{(\mathrm{int})}+b_2^{(\mathrm{int})},\quad \mathrm{agh}\left(b_I^{(\mathrm{int})}\right)=I,\quad I=0,1,2, && (235)\\
n^{\mu} &= n_0^{\mu}+n_1^{\mu}+n_2^{\mu},\quad \mathrm{agh}\left(n_I^{\mu}\right)=I,\quad I=0,1,2. && (236)
\end{aligned}
$$

By projecting the equation (229) on various antighost numbers, we obtain

$$
\begin{aligned}
\gamma b_2^{(\mathrm{int})} &= \partial_{\mu}\left(\frac{1}{2}n_2^{\mu}\right), && (237)\\
\Delta_1^{(\mathrm{int})} &= -2\left(\delta b_2^{(\mathrm{int})}+\gamma b_1^{(\mathrm{int})}\right)+\partial_{\mu}n_1^{\mu}, && (238)\\
\Delta_0^{(\mathrm{int})} &= -2\left(\delta b_1^{(\mathrm{int})}+\gamma b_0^{(\mathrm{int})}\right)+\partial_{\mu}n_0^{\mu}. && (239)
\end{aligned}
$$

The equation (237) can always be replaced, by adding trivial terms, with

$$
\gamma b_2^{(\mathrm{int})}=0. \tag{240}
$$

Looking at $\Delta_1^{(\mathrm{int})}$ given in (232), it results that it can be written like in (238) if

$$
\begin{aligned}
\chi = {} & k\left(1-k\right)\left(\left(\psi^{*}\left(\partial_{\alpha}\psi\right)+\left(\partial_{\alpha}\bar{\psi}\right)\bar{\psi}^{*}\right)\eta_{\beta}\partial^{[\alpha}\eta^{\beta]}\right.\\
&\left.-\frac{1}{16}M^{\alpha\beta}\sigma^{\mu\nu}\partial_{[\alpha}\eta_{\mu]}\partial_{[\beta}\eta_{\nu]}\right),
\end{aligned}
\tag{241}
$$

can be expressed like

$$\chi = \delta\varphi + \gamma\omega + \partial_\mu l^\mu. \tag{242}$$

Supposing that (242) holds and applying δ on it, we infer that

$$\delta\chi = \gamma\left(-\delta\omega\right) + \partial_\mu\left(\delta l^\mu\right). \tag{243}$$

On the other hand, using the concrete expression of χ, we have that

$$\begin{aligned}
\delta\chi &= \gamma\left(\delta\left(k\left(1-k\right)\bar{\psi}\bar{\psi}^*\left(\partial_\mu h^{\mu\beta} - \partial_\beta h\right)\eta_\beta\right)\right) \\
&\quad +\partial_\mu\left(\delta\left(k\left(1-k\right)\bar{\psi}\bar{\psi}^*\eta_\beta\partial^{[\mu}\eta^{\beta]}\right)\right) \\
&\quad +\gamma\left(\mathrm{i}k\left(1-k\right)\bar{\psi}\gamma^\mu\left(\partial_\alpha\psi\right)\left(\frac{1}{2}h_{\mu\beta}\partial^{[\alpha}\eta^{\beta]} - \left(\partial^{[\alpha}h^{\beta]\lambda}\right)\sigma_{\lambda\mu}\eta_\beta\right)\right. \\
&\quad \left. +\frac{\mathrm{i}}{8}k\left(1-k\right)\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\lambda]\mu}\partial_{[\beta}\eta_{\nu]}\sigma^{\lambda\nu}\right) \\
&\quad +\partial_\mu\left(\mathrm{i}k\left(1-k\right)\bar{\psi}\gamma^\mu\left(\partial_\alpha\psi\right)\eta_\beta\partial^{[\alpha}\eta^{\beta]}\right. \\
&\quad \left. +\frac{\mathrm{i}}{16}k\left(1-k\right)\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}\eta_{\lambda]}\partial_{[\beta}\eta_{\nu]}\sigma^{\lambda\nu}\right).
\end{aligned} \tag{244}$$

The right-hand side of (244) can be written like in the right-hand side of (243) if the following conditions are simultaneously satisfied

$$\begin{aligned}
-\delta\omega' &= \bar{\psi}\gamma^\mu\left(\partial_\alpha\psi\right)\left(\frac{1}{2}h_{\mu\beta}\partial^{[\alpha}\eta^{\beta]} - \left(\partial^{[\alpha}h^{\beta]\lambda}\right)\sigma_{\lambda\mu}\eta_\beta\right) \\
&\quad +\frac{1}{8}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}h_{\lambda]\mu}\partial_{[\beta}\eta_{\nu]}\sigma^{\lambda\nu},
\end{aligned} \tag{245}$$

$$\begin{aligned}
\delta l'^\mu &= \bar{\psi}\gamma^\mu\left(\partial_\alpha\psi\right)\eta_\beta\partial^{[\alpha}\eta^{\beta]} \\
&\quad +\frac{1}{16}\bar{\psi}\gamma^\mu\left[\gamma^\alpha,\gamma^\beta\right]\psi\partial_{[\alpha}\eta_{\lambda]}\partial_{[\beta}\eta_{\nu]}\sigma^{\lambda\nu}.
\end{aligned} \tag{246}$$

Since none of the quantities $h_{\mu\beta}$, $\partial^{[\alpha}h^{\beta]\lambda}$, η_β or $\partial^{[\alpha}\eta^{\beta]}$ are δ-exact, the last relations hold if the equations

$$\bar{\psi}\gamma^\mu\left(\partial_\alpha\psi\right) = \delta\Omega^\mu{}_\alpha,\ \bar{\psi}\gamma^\mu\left[\gamma_\alpha,\gamma_\beta\right]\psi = \delta\Gamma^\mu{}_{\alpha\beta} \tag{247}$$

take place simultaneously. Assuming that both the equations (247) are valid, they further give

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\left(\partial_\alpha\psi\right)\right) = \delta\left(\partial_\mu\Omega^\mu{}_\alpha\right), \tag{248}$$

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\left[\gamma_\alpha,\gamma_\beta\right]\psi\right) = \delta\left(\partial_\mu\Gamma^\mu{}_{\alpha\beta}\right). \tag{249}$$

On the other hand, by direct computation we obtain that

$$\partial_\mu \left(\bar{\psi}\gamma^\mu \left(\partial_\alpha \psi\right)\right) = \delta \left(-\mathrm{i}\left(\psi^* \left(\partial_\alpha \psi\right) - \bar{\psi}\left(\partial_\alpha \bar{\psi}^*\right)\right)\right), \tag{250}$$

$$\partial_\mu \left(\bar{\psi}\gamma^\mu \left[\gamma_\alpha, \gamma_\beta\right] \psi\right) = \delta \left(\mathrm{i}M_{\alpha\beta}\right) - 4\bar{\psi}\gamma_{[\alpha}\left(\partial_{\beta]}\psi\right), \tag{251}$$

so the right-hand sides of (250)–(251) cannot be written like in the right-hand sides of (248)–(249). This means that the relations (247) are not valid, and therefore neither are (245)–(246). As a consequence, χ must vanish, and hence we must set

$$k\left(1-k\right) = 0. \tag{252}$$

Using (252), we conclude that

$$k = 1. \tag{253}$$

Inserting (253) in (232), we obtain that

$$\begin{aligned}\Delta_1^{(\mathrm{int})} &= \gamma\left(\left(\psi^*\left(\partial^\alpha \psi\right) + \left(\partial^\alpha \bar{\psi}\right)\bar{\psi}^*\right)\eta^\beta h_{\alpha\beta}\right. \\ &\left. -\frac{1}{16}M^{\alpha\beta}\left(\eta^\sigma \partial_{[\alpha}h_{\beta]\sigma} - \frac{1}{2}h_{\sigma[\alpha}\left(\partial_{\beta]}\eta^\sigma - \partial^\sigma \eta_{\beta]}\right)\right)\right).\end{aligned} \tag{254}$$

Comparing (254) with (238), we find that

$$b_2^{(\mathrm{int})} = 0, \tag{255}$$

$$\begin{aligned}b_1^{(\mathrm{int})} &= -\frac{1}{2}\left(\psi^*\left(\partial^\alpha \psi\right) + \left(\partial^\alpha \bar{\psi}\right)\bar{\psi}^*\right)\eta^\beta h_{\alpha\beta} + \frac{1}{32}\left(\bar{\psi}\left[\gamma^\alpha, \gamma^\beta\right]\bar{\psi}^*\right. \\ &\left. -\psi^*\left[\gamma^\alpha, \gamma^\beta\right]\psi\right)\left(\eta^\sigma \partial_{[\alpha}h_{\beta]\sigma} - \frac{1}{2}h_{\sigma[\alpha}\left(\partial_{\beta]}\eta^\sigma - \partial^\sigma \eta_{\beta]}\right)\right).\end{aligned} \tag{256}$$

Substituting (253) in (233) and using (256), we deduce

$$\begin{aligned}\Delta_0^{(\mathrm{int})} + 2\delta b_1^{(\mathrm{int})} &= \partial_\mu n_0^\mu + \gamma\left(\frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu \partial^\nu \psi\left(hh_{\mu\nu} - \frac{3}{2}h_{\mu\sigma}h_\nu^\sigma\right)\right. \\ &+\frac{1}{2}\left(\bar{\psi}\mathrm{i}\gamma^\mu\left(\partial_\mu \psi\right) - m\bar{\psi}\psi\right)\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}h^2\right) \\ &+\frac{\mathrm{i}}{16}\bar{\psi}\gamma^\mu\left[\gamma^\alpha, \gamma^\beta\right]\psi\left(h\partial_{[\alpha}h_{\beta]\mu} - h_\mu^\sigma \partial_{[\alpha}h_{\beta]\sigma}\right. \\ &\left.+h_\alpha^\sigma\left(2\partial_{[\beta}h_{\sigma]\mu} + \partial_\mu h_{\beta\sigma}\right)\right) \\ &-\left(f\left(\bar{\psi}, \psi\right) + \left(\partial_\mu \bar{\psi}\right)g_1^\mu\left(\bar{\psi}, \psi\right) + g_2^\mu\left(\bar{\psi}, \psi\right)\left(\partial_\mu \psi\right)\right)h\end{aligned}$$

$$
\begin{aligned}
&+h_{\mu\nu}\left(\left(\partial^{\nu}\bar{\psi}\right)g_1^{\mu}\left(\bar{\psi},\psi\right)+g_2^{\mu}\left(\bar{\psi},\psi\right)\left(\partial^{\nu}\psi\right)\right)\\
&+\frac{1}{8}\left(\frac{\partial^R f}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L f}{\partial\bar{\psi}}\right)\partial_{[\mu}\eta_{\nu]}\\
&+\frac{1}{8}\partial_{[\mu}\eta_{\nu]}\left(\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\left(\partial_{\rho}g_1^{\rho}\right)-\left(\partial_{\rho}g_2^{\rho}\right)\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\right.\\
&+\frac{\partial^R\left(\left(\partial_{\rho}\bar{\psi}\right)g_1^{\rho}+g_2^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\\
&-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L\left(\left(\partial_{\rho}\bar{\psi}\right)g_1^{\rho}+g_2^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\bar{\psi}}\\
&\left.-4\left(\left(\partial^{[\mu}\bar{\psi}\right)g_1^{\nu]}-g_2^{[\mu}\left(\partial^{\nu]}\psi\right)\right)\right). \qquad (257)
\end{aligned}
$$

The right-hand side of (257) can be written like in (239) if

$$
\begin{aligned}
&\frac{1}{8}\left(\frac{\partial^R f}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L f}{\partial\bar{\psi}}\right)\partial_{[\mu}\eta_{\nu]}\\
&+\frac{1}{8}\partial_{[\mu}\eta_{\nu]}\left(\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\left(\partial_{\rho}g_1^{\rho}\right)-\left(\partial_{\rho}g_2^{\rho}\right)\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\right.\\
&+\frac{\partial^R\left(\left(\partial_{\rho}\bar{\psi}\right)g_1^{\rho}+g_2^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\\
&-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L\left(\left(\partial_{\rho}\bar{\psi}\right)g_1^{\rho}+g_2^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\bar{\psi}}\\
&\left.-4\left(\left(\partial^{[\mu}\bar{\psi}\right)g_1^{\nu]}-g_2^{[\mu}\left(\partial^{\nu]}\psi\right)\right)\right)=\gamma\theta+\partial_{\mu}\rho^{\mu}. \qquad (258)
\end{aligned}
$$

The term $\frac{1}{8}\left(\frac{\partial^R f}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L f}{\partial\bar{\psi}}\right)\partial_{[\mu}\eta_{\nu]}$ is neither γ-exact nor a total derivative (as $f\left(\bar{\psi},\psi\right)$ has no derivatives), and hence we must require that

$$
\frac{\partial^R f}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^L f}{\partial\bar{\psi}}=0. \qquad (259)
$$

The solution to (259) reads as

$$
f\left(\bar{\psi},\psi\right)=M\left(\bar{\psi}\psi\right), \qquad (260)
$$

where M is a polynomial in $\bar{\psi}\psi$. With (259) at hand, the equation (258) becomes

$$
\frac{1}{8}\left(\partial_{[\mu}\eta_{\nu]}\right)\Pi^{\mu\nu}=\gamma\theta+\partial_{\mu}\rho^{\mu}, \qquad (261)
$$

where

$$\begin{aligned}
\Pi^{\mu\nu} &\equiv \bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\left(\partial_{\rho}g_{1}^{\rho}\right)-\left(\partial_{\rho}g_{2}^{\rho}\right)\left[\gamma^{\mu},\gamma^{\nu}\right]\psi \\
&-4\left(\left(\partial^{[\mu}\bar{\psi}\right)g_{1}^{\nu]}-g_{2}^{[\mu}\left(\partial^{\nu]}\psi\right)\right) \\
&+\frac{\partial^{R}\left(\left(\partial_{\rho}\bar{\psi}\right)g_{1}^{\rho}+g_{2}^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi \\
&-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^{L}\left(\left(\partial_{\rho}\bar{\psi}\right)g_{1}^{\rho}+g_{2}^{\rho}\left(\partial_{\rho}\psi\right)\right)}{\partial\bar{\psi}}. \qquad (262)
\end{aligned}$$

The left-hand side of (261) is γ-exact modulo d if there exists a real, bosonic function, involving only the undifferentiated Dirac fields, $M^{\rho\mu\nu}$, which is antisymmetric in its last two indices

$$M^{\rho\mu\nu}=-M^{\rho\nu\mu}, \qquad (263)$$

such that

$$\Pi^{\mu\nu}=\partial_{\rho}M^{\rho\mu\nu}. \qquad (264)$$

From (262), we observe that the existence of such functions $M^{\rho\mu\nu}$ is controlled by the functions g_{1}^{μ} and g_{2}^{μ}. The most general form of the functions g_{1}^{μ} is

$$g_{1}^{\mu}=\psi g^{\mu}+\sum_{n=1}^{4}\gamma^{\nu_{1}}\cdots\gamma^{\nu_{n}}\psi g_{\nu_{1}\ldots\nu_{n}}^{\mu}, \qquad (265)$$

where g^{μ} and $g_{\nu_{1}\ldots\nu_{n}}^{\mu}$ are some real, bosonic functions in the undifferentiated Dirac fields. Now, from (109) it results that

$$g_{2}^{\mu}=g^{\mu}\bar{\psi}+\sum_{n=1}^{4}g_{\nu_{1}\ldots\nu_{n}}^{\mu}\bar{\psi}\gamma^{\nu_{n}}\cdots\gamma^{\nu_{1}}. \qquad (266)$$

Inserting (265)–(266) in (262), we arrive at

$$\begin{aligned}
\Pi^{\mu\nu} &= \left(\partial_{\rho}\left(\bar{\psi}\psi\right)\right)\left(\frac{\partial^{R}g^{\rho}}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^{L}g^{\rho}}{\partial\bar{\psi}}-4\sigma^{\rho[\mu}g^{\nu]}\right) \\
&+\partial_{\rho}\left(-\sum_{n=1}^{4}g_{\nu_{1}\ldots\nu_{n}}^{\rho}\left(\bar{\psi}\gamma^{\nu_{n}}\cdots\gamma^{\nu_{1}}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\gamma^{\nu_{1}}\cdots\gamma^{\nu_{n}}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi\right)\right) \\
&+\sum_{n=1}^{4}\left(\frac{\partial^{R}g_{\nu_{1}\ldots\nu_{n}}^{\rho}}{\partial\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi-\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\frac{\partial^{L}g_{\nu_{1}\ldots\nu_{n}}^{\rho}}{\partial\bar{\psi}}-4\sigma^{\rho[\mu}g_{\nu_{1}\ldots\nu_{n}}^{\nu]}\right)\times
\end{aligned}$$

$$\times\left(\left(\partial_\rho\bar{\psi}\right)\gamma^{\nu_1}\cdots\gamma^{\nu_n}\psi+\bar{\psi}\gamma^{\nu_n}\cdots\gamma^{\nu_1}\left(\partial_\rho\psi\right)\right)$$
$$+4\sum_{n=1}^{4}g^{\rho}_{\nu_1\ldots\nu_n}\sum_{k=1}^{n}\left(\left(\partial_\rho\bar{\psi}\right)\gamma^{\nu_1}\cdots\gamma^{\nu_{k-1}}\sigma^{\nu_k[\mu}\gamma^{\nu]}\gamma^{\nu_{k+1}}\cdots\gamma^{\nu_n}\psi\right.$$
$$\left.+\bar{\psi}\gamma^{\nu_n}\cdots\gamma^{\nu_{k+1}}\sigma^{\nu_k[\mu}\gamma^{\nu]}\gamma^{\nu_{k-1}}\cdots\gamma^{\nu_1}\left(\partial_\rho\psi\right)\right). \tag{267}$$

The right-hand side of (267) is of the form $\partial_\rho M^{\rho\mu\nu}$ if

$$\frac{\partial^R g^\rho}{\partial\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi-\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\frac{\partial^L g^\rho}{\partial\bar{\psi}}-4\sigma^{\rho[\mu}g^{\nu]}=C^{\rho\mu\nu}, \tag{268}$$
$$n=1,\ g^{\rho}_{\nu_1}=\delta^{\rho}_{\nu_1}Q\left(\bar{\psi}\psi\right), \tag{269}$$

where $C^{\rho\mu\nu}=-C^{\rho\nu\mu}$ are some constants (and not some 4×4 matrices) and Q is an arbitrary polynomial in $\bar{\psi}\psi$. Since in four spacetime dimensions there are no such constants, they must vanish, which imply that

$$\frac{\partial^R g^\rho}{\partial\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi-\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\frac{\partial^L g^\rho}{\partial\bar{\psi}}-4\sigma^{\rho[\mu}g^{\nu]}=0. \tag{270}$$

The solution to the last equation reads as

$$g^\rho=\bar{\psi}\gamma^\rho\psi N\left(\bar{\psi}\psi\right), \tag{271}$$

where N is an arbitrary polynomial in $\bar{\psi}\psi$. In consequence, $\Pi^{\mu\nu}$ expressed by (262) can be written like in (264) if the functions (265)–(266) are of the form

$$g_1^\mu = \psi\left(\bar{\psi}\gamma^\mu\psi\right)N\left(\bar{\psi}\psi\right)+\gamma^\mu\psi Q\left(\bar{\psi}\psi\right), \tag{272}$$
$$g_2^\mu = \left(\bar{\psi}\gamma^\mu\psi\right)N\left(\bar{\psi}\psi\right)\bar{\psi}+\bar{\psi}\gamma^\mu Q\left(\bar{\psi}\psi\right). \tag{273}$$

By means of (272)–(273) we deduce

$$\left(\partial_\mu\bar{\psi}\right)g_1^\mu\left(\bar{\psi},\psi\right)+g_2^\mu\left(\bar{\psi},\psi\right)\left(\partial_\mu\psi\right)=\partial_\mu\left(\bar{\psi}\gamma^\mu\psi P\left(\bar{\psi}\psi\right)\right)$$
$$+s\left(\mathrm{i}\left(\psi^*\psi-\bar{\psi}\bar{\psi}^*\right)\left(P\left(\bar{\psi}\psi\right)-Q\left(\bar{\psi}\psi\right)\right)\right), \tag{274}$$

where P is a polynomial in $\bar{\psi}\psi$ defined by $N\left(\bar{\psi}\psi\right)=dP\left(\bar{\psi}\psi\right)/d\left(\bar{\psi}\psi\right)$. The relation (274) shows that the last two terms from (108) produce a trivial deformation, which can always be removed by setting

$$g_1^\mu\left(\bar{\psi},\psi\right)=0,\ g_2^\mu\left(\bar{\psi},\psi\right)=0. \tag{275}$$

Then, with the help of (260) it follows that

$$a_0^{(\mathrm{Dirac})} = M\left(\bar{\psi}\psi\right). \tag{276}$$

Inserting (260) and (275) in (257), we finally find that the interacting lagrangian at order two in the coupling constant takes the form

$$\begin{aligned} b_0^{(\mathrm{int})} = & -\frac{\mathrm{i}}{4}\bar{\psi}\gamma^\mu\left(\partial^\nu\psi\right)\left(hh_{\mu\nu} - \frac{3}{2}h_{\mu\sigma}h_\nu^\sigma\right) - \frac{1}{4}\left(\bar{\psi}\mathrm{i}\gamma^\mu\left(\partial_\mu\psi\right) - m\bar{\psi}\psi\right) \times \\ & \times\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}h^2\right) - \frac{\mathrm{i}}{32}\bar{\psi}\gamma^\mu\left[\gamma^\alpha, \gamma^\beta\right]\psi\left(h\partial_{[\alpha}h_{\beta]\mu}\right. \\ & \left.-h_\mu^\sigma\partial_{[\alpha}h_{\beta]\sigma} + h_\alpha^\sigma\left(2\partial_{[\beta}h_{\sigma]\mu} + \partial_\mu h_{\beta\sigma}\right)\right) + \frac{1}{2}M\left(\bar{\psi}\psi\right)h. \end{aligned} \tag{277}$$

The formulas (255), (256) and (277) reveal the full, interacting, second-order deformation of the solution to the master equation.

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